Outline

Propositional Logic
- Syntax
- Semantics
- Propositional Satisfiability
- Clausal Forms
- Clausal Form and Definitional Transformation

Propositional logic: syntax

Assume a countable set of boolean variables.
Propositional formula:

- Every boolean variable is a formula, also called atomic formula, or simply atom.
- $\top$ and $\perp$ are formulas.
- If $A_1, \ldots, A_n$ are formulas, where $n \geq 2$, then $(A_1 \wedge \ldots \wedge A_n)$ and $(A_1 \vee \ldots \vee A_n)$ are formulas.
- If A is a formula, then $\neg A$ is a formula.
- If A and B are formulas, then $(A \rightarrow B)$ and $(A \leftrightarrow B)$ are formulas.

The symbols $\top, \perp, \wedge, \vee, \neg, \rightarrow, \leftrightarrow$ are called connectives.

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Connectives

Connective	Name	Priority
$\top$	verum	
$\bot$	falsum	
$\neg$	negation	4
$\wedge$	conjunction	3
$\vee$	disjunction	3
$\rightarrow$	implication	2
$\leftrightarrow$	equivalence	1

Parsing Formulas

We normally omit parenthesis in mathematical expressions and use priorities to disambiguate them.

For example, in arithmetic we know that the expression

$$x \cdot y + 2 \cdot z$$

is equivalent to

$$(x \cdot y) + (2 \cdot z),$$

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Parsing: Example

Let's parse $\neg A \land B \to C \lor D \leftrightarrow E$.

Connective	Priority
$\top$	
$\bot$	
$\neg$	4
$\land$	3
$\lor$	3
$\to$	2
$\leftrightarrow$	1

Inside-out (starting with the highest priority connectives):

$$(((\neg A) \land B) \to (C \lor D)) \leftrightarrow E.$$

Outside-in (starting with the lowest priority connectives):

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Semantics, Interpretation

Consider an arithmetical expression, for example

$$x \cdot y + 2 \cdot z.$$

In arithmetic the meaning of expressions with variables is defined as follows.

Take a mapping from variables (integer) values, for example

$$\{x \mapsto 1, y \mapsto 7, z \mapsto -3\}.$$

Then, under this mapping the expression has the value 1. In other words, when we interpret variables as values, we can compute the value of the expression.

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Semantics, Interpretation

Likewise, the semantics of propositional formulas can be defined by assigning boolean values to variables.

- A boolean value, also called a truth value, is either true (denoted 1) or false (denoted 0).
- An interpretation for a set P of boolean variables is a mapping $I : P \to \{1, 0\}$.
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Interpreting formulas

Extend I to all formulas:

1. $I(\top) = 1$ and $I(\bot) = 0$.
2. $I(A_1 \wedge \ldots \wedge A_n) = 1$ if and only if $I(A_i) = 1$ for all i.
3. $I(A_1 \vee \ldots \vee A_n) = 1$ if and only if $I(A_i) = 1$ for some i.
4. $I(\neg A) = 1$ if and only if $I(A) = 0$.
5. $I(A_1 \rightarrow A_2) = 1$ if and only if $I(A_1) = 0$ or $I(A_2) = 1$.
6. $I(A_1 \leftrightarrow A_2) = 1$ if and only if $I(A_1) = I(A_2)$.

Operation tables

$I(A_1 \vee A_2) = 1$ if and only if $I(A_1) = 1$ or $I(A_2) = 1$.
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0	0	0

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Satisfiability, validity

- If $I(A) = 1$, then we say that the formula A is true in I and that I satisfies A and that I is a model of A, denoted by $I \models A$.
- If $I(A) = 0$, then we say that the formula A is false in I.
- A is satisfiable (valid) if it is true in some (every) interpretation.
- Two formulas A and B are called equivalent, denoted by $A \equiv B$ if they have the same models.

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Examples

$A \rightarrow A$ and $A \vee \neg A$ are valid for all formulas A.

Evidently, every valid formula is also satisfiable.

$A \wedge \neg A$ is unsatisfiable.

Formula p, where p is a boolean variable, is satisfiable but not valid.

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Examples: equivalences

For all formulas A and B, the following equivalences hold.

$$\begin{aligned}
A \rightarrow \bot &\equiv \neg A; && (1)\\
\top \rightarrow A &\equiv A; && (2)\\
A \rightarrow B &\equiv \neg(A \land \neg B); && (3)\\
A \land B &\equiv \neg(\neg A \lor \neg B); && (4)\\
A \lor B &\equiv \neg A \rightarrow B. && (5)
\end{aligned}$$

Connections between these notions

1. A formula A is valid if and only if $\neg A$ is unsatisfiable.
2. A formula A is satisfiable if and only if $\neg A$ is not valid.
3. A formula A is valid if and only if A is equivalent to $\top$.
4. Formulas A and B are equivalent if and only if the formula $A \leftrightarrow B$ is valid.

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Equivalent replacement

We denote by $A[B]$ a formula A with a fixed occurrence of a subformula B. If we use this notation we can also write $A[B']$ to denote the formula obtained from A by replacing this occurrence of B by B'.

Lemma (Equivalent Replacement)

Let I be an interpretation and $I \models A_1 \leftrightarrow A_2$. Then $I \models B[A_1] \leftrightarrow B[A_2]$.

Theorem (Equivalent Replacement)

Let $A_1 \equiv A_2$. Then Then $B[A_1] \equiv B[A_2]$.

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Given a propositional formula A, check wheter it is satisfiable or not.

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Russian spy puzzle

There are three persons: Stirlitz, Müller, and Eismann. It is known that exactly one of them is Russian, while the other two are Germans. Moreover, every Russian must be a spy.

When Stirlitz meets Müller in a corridor, he makes the following joke: “you know, Müller, you are as German as I am Russian”. It is known that Stirlitz always tells the truth when he is joking.

We have to establish that Eismann is not a Russian spy.

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Formalisation in propositional logic

Introduce propositional variables XY with the following meaning in mind:

$X \in \{R, G, S\}$ (denoting Russian, German, Spy)
$Y \in \{S, M, E\}$ (denoting Stirlitz, Müller, Eismann)

For example,

SE : Eismann is a Spy
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Why satisfiability?

A formula A is a logical consequence of formulas $A_1, \ldots, A_n$, or follows from $A_1, \ldots, A_n$, if every model of $A_1, \ldots, A_n$ is also a model of A.

Note that A is not a logical consequence of $A_1, \ldots, A_n$ if and only if the set of formulas $A_1, \ldots, A_n, \neg A$ is satisfiable.

We have to determine whether the fact that Eismann is not a Russian spy follows from the conditions of the puzzle.

Therefore, the problem of solving the puzzle is an instance of the satisfaibility problem.

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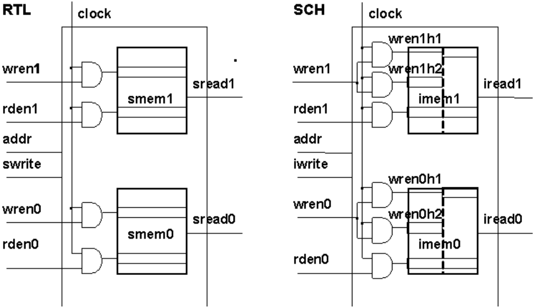
Circuit Equivalence

Given two circuits, check if they are equivalent. For example:



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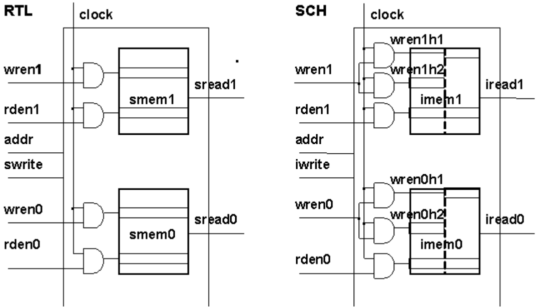

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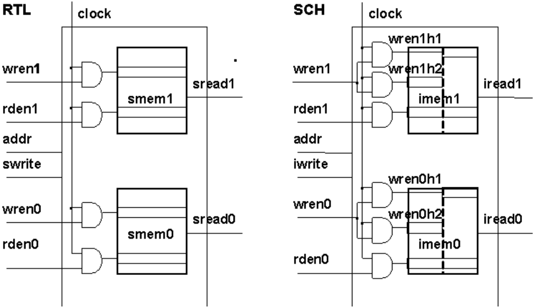

Every circuit is, in fact, a propositional formula.

We know that equivalence-checking for propositional formulas can be reduced to unsatisfiability-checking.

Satisfiability?

Satisfiability checking is a combinatorial problem that is

- easy to formulate;
- hard to solve;
- NP-complete;
- has many algorithms (but only one is commonly used).

Literal, clause

- Literal: either an atom p (positive literal) or its negation $\neg p$ (negative literal).
- The complementary literal to L:

$$\overline{L} \stackrel{\text{def}}{\Leftrightarrow} \begin{cases} \neg L, & \text{if } L \text{ is positive;} \\ p, & \text{if } L \text{ has the form } \neg p. \end{cases}$$

  In other words, p and $\neg p$ are complementary.
- Clause: a disjunction $L_1 \vee \ldots \vee L_n$, $n \geq 0$ of literals.
  - Empty clause, denoted by $\square$: $n = 0$ (the empty clause is false in every interpretation).
  - Unit clause: $n = 1$.
  - Horn clause: a clause with at most one positive literal.

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CNF

- A formula A is in conjunctive normal form, or simply CNF, if it is either $\top$, or $\bot$, or a conjunction of disjunctions of literals:

$$A = \bigwedge_i \bigvee_j L_{i,j}.$$

  (That is, a conjunction of clauses.)
- A formula B is called a conjunctive normal form of a formula A if B is equivalent to A and B is in conjunctive normal form.

Satisfiability on CNF

- An interpretation I satisfies a formula in CNF

$$A = \bigwedge_i \bigvee_j L_{i,j}.$$

  if and only if it satisfies every clause

$$\bigvee_j L_{i,j}.$$

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CNF transformation

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\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), \\
A \rightarrow B & \Rightarrow & \neg A \vee B, \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & \begin{array}{cc} (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\ \ldots & \wedge \\ (A_m \vee B_1 \vee \ldots \vee B_n). & \end{array}
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A formula to which no rewrite rule is applicable

- contains no $\leftrightarrow$;
- contains no $\rightarrow$;
- may only contain $\neg$ applied to atoms;
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CNF, example

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow$$
$$\neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow$$
$$\neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge r \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow$$
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(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
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\end{array}$$

CNF, example

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\begin{array}{l}
\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow \\
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\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow \\
(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow \\
(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow \\
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\neg\neg A & \Rightarrow & A, & \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & \quad (A_1 \lor B_1 \lor \ldots \lor B_n) & \land \\
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$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r)) \Rightarrow$
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$$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \wedge (p \wedge q \to r)) \vee (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \wedge (p \wedge q \to r)) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg(\neg p \vee r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg\neg p \wedge r \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$
$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \to B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}
$$

CNF, example

$$
\begin{array}{l}
\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow \\
\neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow \\
\neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge r \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \\
(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r
\end{array}
$$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \rightarrow B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}
$$

CNF, example

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow$$
$$\neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow$$
$$\neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge r \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$
$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), \\
A \rightarrow B & \Rightarrow & \neg A \vee B, \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) \quad \wedge \\
 & & \ldots \quad \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n).
\end{array}$$

CNF, example

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \land q) \lor r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), \\
A \to B & \Rightarrow & \neg A \lor B, \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) \quad \land \\
 & & \quad \ldots \quad \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n).
\end{array}$$

CNF, example

$$
\begin{array}{l}
\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r)) \Rightarrow \\
\neg(\neg((p \to q) \wedge (p \wedge q \to r)) \vee (p \to r)) \Rightarrow \\
\neg\neg((p \to q) \wedge (p \wedge q \to r)) \wedge \neg(p \to r) \Rightarrow \\
(p \to q) \wedge (p \wedge q \to r) \wedge \neg(p \to r) \Rightarrow \\
(p \to q) \wedge (p \wedge q \to r) \wedge \neg(\neg p \vee r) \Rightarrow \\
(p \to q) \wedge (p \wedge q \to r) \wedge \neg\neg p \wedge r \Rightarrow \\
(p \to q) \wedge (p \wedge q \to r) \wedge p \wedge \neg r \Rightarrow \\
(p \to q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow \\
(p \to q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \\
(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r
\end{array}
$$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \to B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}
$$

CNF, example

$$
\begin{array}{l}
\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow \\
\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow \\
\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow \\
(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow \\
(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow \\
(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow \\
(p \to q) \land (p \land q \to r) \land p \land \neg r \Rightarrow \\
(p \to q) \land (\neg(p \land q) \lor r) \land p \land \neg r \Rightarrow \\
(p \to q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r \\
(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r
\end{array}
$$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), & \\
A \to B & \Rightarrow & \neg A \lor B, & \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, & \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) & \land \\
 & & \ldots & \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n). &
\end{array}
$$

CNF, example

$$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \wedge (p \wedge q \to r)) \vee (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \wedge (p \wedge q \to r)) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg(\neg p \vee r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg\neg p \wedge r \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$
$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), \\
A \to B & \Rightarrow & \neg A \vee B, \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & \begin{array}{cc} (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\ \cdots & \wedge \\ (A_m \vee B_1 \vee \ldots \vee B_n). & \end{array}
\end{array}$$

CNF, example

$$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \wedge (p \wedge q \to r)) \vee (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \wedge (p \wedge q \to r)) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg(\neg p \vee r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg\neg p \wedge r \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$
$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), \\
A \to B & \Rightarrow & \neg A \vee B, \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & \begin{array}{cc} (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\ \ldots & \wedge \\ (A_m \vee B_1 \vee \ldots \vee B_n). & \end{array}
\end{array}$$

CNF, example

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \land q) \lor r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), \\
A \to B & \Rightarrow & \neg A \lor B, \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) \quad \land \\
 & & \ldots \quad \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n).
\end{array}$$

CNF, example

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \land q) \lor r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

$$\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), & \\
A \to B & \Rightarrow & \neg A \lor B, & \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, & \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) & \land \\
 & & \ldots & \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n). &
\end{array}$$

CNF, example

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \land q) \lor r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), & \\
A \to B & \Rightarrow & \neg A \lor B, & \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, & \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) & \land \\
 & & \ldots & \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n). &
\end{array}
$$

CNF, example

$$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \wedge (p \wedge q \to r)) \vee (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \wedge (p \wedge q \to r)) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg(\neg p \vee r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg\neg p \wedge r \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \to q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$
$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

$$\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \to B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}$$

CNF and satisfiability

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\dots$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

Therefore, the formula

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r))$$

has the same models as the set consisting of four clauses

$$\begin{array}{l} \neg p \lor q \\ \neg p \lor \neg q \lor r \\ p \\ \neg r \end{array}$$

The CNF transformation allows one to reduce the satisfiability problem for formulas to the satisfiability problem for sets of clauses.

CNF and satisfiability

$$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r)) \Rightarrow$$
$$\cdots$$
$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

Therefore, the formula

$$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r))$$

has the same models as the set consisting of four clauses

$$\begin{array}{l} \neg p \vee q \\ \neg p \vee \neg q \vee r \\ p \\ \neg r \end{array}$$

The CNF transformation allows one to reduce the satisfiability problem for formulas to the satisfiability problem for sets of clauses.

CNF and satisfiability

$$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r)) \Rightarrow$$
$$\dots$$
$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

Therefore, the formula

$$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r))$$

has the same models as the set consisting of four clauses

$$\begin{array}{l} \neg p \vee q \\ \neg p \vee \neg q \vee r \\ p \\ \neg r \end{array}$$

The CNF transformation allows one to reduce the satisfiability problem for formulas to the satisfiability problem for sets of clauses.

Problem

Compute the CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))).$$

$$\begin{aligned}
&p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \Rightarrow\\
&(\neg p_1 \vee (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \wedge\\
&(p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))\\
&(\neg p_1 \vee ((\neg p_2 \vee (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))\wedge\\
&\qquad\qquad (p_2 \vee \neg(p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))\wedge\\
&(p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))
\end{aligned}$$

If we continue, the formula will grow exponentially.

Problem

Compute the CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))).$$

$$\begin{array}{l}
p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \Rightarrow \\
(\neg p_1 \lor (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \land \\
(p_1 \lor \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \Rightarrow \\
(\neg p_1 \lor \quad ((\neg p_2 \lor (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))\land \\
\qquad\qquad (p_2 \lor \neg(p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))))\land \\
(p_1 \lor \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))
\end{array}$$

If we continue, the formula will grow exponentially.

Problem

Compute the CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))).$$

$$\begin{array}{l}
p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \Rightarrow \\
(\neg p_1 \vee (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \wedge \\
(p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \Rightarrow \\
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If we continue, the formula will grow exponentially.

Problem

Compute the CNF of

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If we continue, the formula will grow exponentially.

CNF is exponential

There are formulas for which the shortest CNF has exponential size.

Is there any way to avoid exponential blowup?

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Using so-called naming or definition introduction.

- Take a non-trivial subformula A.
- Introduce a new name n for it. A name is a new propositional variable.
- Add a formula stating that n is equivalent to A (definition for n).

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\begin{array}{l}
p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \\
n \leftrightarrow (p_5 \leftrightarrow p_6)
\end{array}
$$

- Replace the subformula by its name:

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p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow n))) \\
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The new set of two formulas has the same models as the original one if we restrict ourselves to the original set of variables $\{p_1, \ldots, p_6\}$.
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After several steps

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The conversion of the original formula to CNF introduces 32 copies of p_6.

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Clausal Form

- Clausal form of a formula A: a set of clauses which is satisfiable if and only if A is satisfiable.
- Clausal form of a set S of formulas: a set of clauses which is satisfiable if and only if so is S.

We can require even more: that A and S have the same models in the language of A.

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Definitional Clause Form Transformation

This algorithm converts a formula A into a set of clauses S such that S is a clausal normal form of A.

If A has the form $C_1 \wedge \ldots \wedge C_n$, where $n \geq 1$ and each C_i is a clause, then $S \overset{\text{def}}{\Leftrightarrow} \{C_1, \ldots, C_n\}$.

Otherwise, introduce a name for each subformula B of A such that B is not a literal and use this name instead of the formula.

Example

	subformula	definition	clauses
			n_1
n_1	$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \vee \neg n_2$ $n_1 \vee n_2$
n_2	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	$n_2 \leftrightarrow (n_3 \to n_7)$	$\neg n_2 \vee \neg n_3 \vee n_7$ $n_3 \vee n_2$ $\neg n_7 \vee n_2$
n_3	$(p \to q) \wedge (p \wedge q \to r)$	$n_3 \leftrightarrow (n_4 \wedge n_5)$	$\neg n_3 \vee n_4$ $\neg n_3 \vee n_5$ $\neg n_4 \vee \neg n_5 \vee n_3$
n_4	$p \to q$	$n_4 \leftrightarrow (p \to q)$	$\neg n_4 \vee \neg p \vee q$ $p \vee n_4$ $\neg q \vee n_4$
n_5	$p \wedge q \to r$	$n_5 \leftrightarrow (n_6 \to r)$	$\neg n_5 \vee \neg n_6 \vee r$ $n_6 \vee n_5$ $\neg r \vee n_5$
n_6	$p \wedge q$	$n_6 \leftrightarrow (p \wedge q)$	$\neg n_6 \vee p$ $\neg n_6 \vee q$ $\neg p \vee \neg q \vee n_6$
n_7	$p \to r$	$n_7 \leftrightarrow (p \to r)$	$\neg n_7 \vee \neg p \vee r$ $p \vee n_7$ $\neg r \vee n_7$