Evaluating a formula

Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$.

$$I_{10} \models \forall p \exists q (p \leftrightarrow q) \Leftrightarrow \boxed{\begin{array}{ll} I_{00} \models \exists q(p \leftrightarrow q) & \\ & \text{and} \\ I_{10} \models \exists q(p \leftrightarrow q) & \end{array}} \Leftrightarrow \boxed{\begin{array}{ll} \boxed{\begin{array}{ll} I_{00} \models p \leftrightarrow q & \\ & \text{or} \\ I_{01} \models p \leftrightarrow q & \end{array}} & \\ & \text{and} \\ \boxed{\begin{array}{ll} I_{10} \models p \leftrightarrow q & \\ & \text{or} \\ I_{11} \models p \leftrightarrow q & \end{array}} & \end{array}}$$

Evaluating a formula

$$I_{--} \models \forall p \exists q (p \leftrightarrow q)$$

$$\wedge$$

$$I_{0-} \models \exists q (p \leftrightarrow q) \qquad I_{1-} \models \exists q (p \leftrightarrow q)$$

$$\vee \qquad \vee$$

$$I_{00} \models p \leftrightarrow q \qquad I_{01} \models p \leftrightarrow q \qquad I_{10} \models p \leftrightarrow q \qquad I_{11} \models p \leftrightarrow q$$

Evaluating a formula

Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$. Use wildcards $\_$ to denote "any" boolean value.

$$I_{\_\_} \models \forall p \exists q (p \leftrightarrow q) \Leftrightarrow \boxed{\begin{array}{ll} I_{0\_} \models \exists q (p \leftrightarrow q) & \\ & \text{and} \\ I_{1\_} \models \exists q (p \leftrightarrow q) & \end{array}} \Leftrightarrow \boxed{\begin{array}{ll} \boxed{\begin{array}{ll} I_{00} \models p \leftrightarrow q & \\ & \text{or} \\ I_{01} \models p \leftrightarrow q & \end{array}} & \\ & \text{and} \\ \boxed{\begin{array}{ll} I_{10} \models p \leftrightarrow q & \\ & \text{or} \\ I_{11} \models p \leftrightarrow q & \end{array}} & \end{array}}$$

Evaluating a formula

The variables p and q are bound by quantifiers $\forall p$ and $\exists q$, so the value of the formula does not depend on these variables.

$$I_{--} \models \forall p \exists q (p \leftrightarrow q) \Leftrightarrow \boxed{\begin{array}{ll} I_{0-} \models \exists q (p \leftrightarrow q) & \\ & \text{and} \\ I_{1-} \models \exists q (p \leftrightarrow q) & \end{array}} \Leftrightarrow \boxed{\begin{array}{ll} \boxed{\begin{array}{ll} I_{00} \models p \leftrightarrow q & \\ & \text{or} \\ I_{01} \models p \leftrightarrow q & \end{array}} & \\ & \text{and} \\ \boxed{\begin{array}{ll} I_{10} \models p \leftrightarrow q & \\ & \text{or} \\ I_{11} \models p \leftrightarrow q & \end{array}} & \end{array}}$$

Subformula

Propositional formulas:

- ▷ The formulas $F_1, \ldots, F_n$ are the immediate subformulas of the formulas $F_1 \wedge \ldots \wedge F_n$ and $F_1 \vee \ldots \vee F_n$.
- ▷ The formulas F is the immediate subformula of the formula $\neg F$.
- ▷ The formulas F_1, F_2 are the immediate subformulas of the formulas $F_1 \rightarrow F_2$ and $F_1 \leftrightarrow F_2$.
- ▷ ...

Quantified boolean formulas:

- ▷ The formula F_1 is the immediate subformula of the formulas $\forall p F_1$ and $\exists p F_1$.

Positions

Let $F|_\pi = G$.

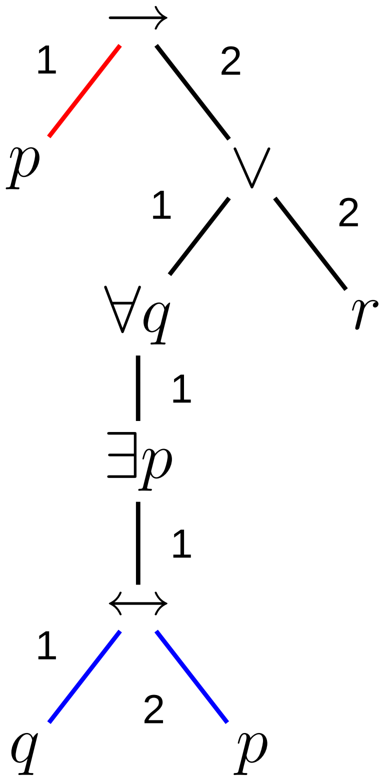
Propositional formulas:

- ▷ If G has the form $G_1 \wedge \ldots \wedge G_n$ or $G_1 \vee \ldots \vee G_n$, then for all $i \in \{1, \ldots, n\}$ the position $\pi.i$ is a position in F and $pol(F, \pi.i) \stackrel{\text{def}}{=} pol(F, \pi)$.
- ▷ If G has the form $\neg G_1$, then $\pi.1$ is a position in F, $F|_{\pi.1} \stackrel{\text{def}}{=} G_1$ and $pol(F, \pi.1) \stackrel{\text{def}}{=} -pol(F, \pi)$.
- ▷ ...

Quantified boolean formulas:

- ▷ If G has the form $\forall p G_1$ or $\exists p G_1$, then $\pi.1$ is a position in F, $F|_{\pi.1} \stackrel{\text{def}}{=} G_1$ and $pol(F, \pi.1) \stackrel{\text{def}}{=} pol(F, \pi)$.

Example



$$F|_{2.1} = \forall q \exists p (q \leftrightarrow p)$$

$$F|_{2.1.1.1.1} = q$$

Free and bound occurrences of variables

Let p be a boolean variable and $F|_\pi = p$.

- The occurrence of p at the position π in F is bound if π can be represented as a concatenation of two strings $\pi_1\pi_2$ such that $F|_{\pi_1}$ has the form $\forall pG$ or $\exists pG$ for some G.

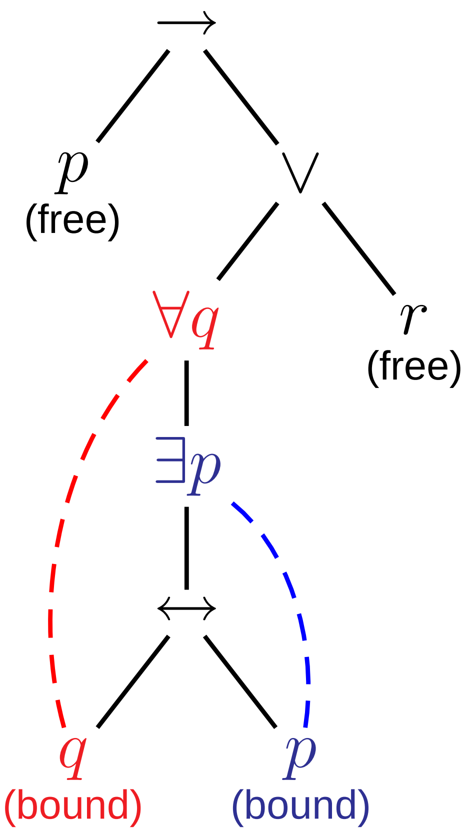
  In other words, a bound occurence of p is an occurrence in the range of $\forall p$ or $\exists p$.

- Free occurrence: not bound.

- Free (bound) variable of a formula: a variable with at least one free (bound) occurrence.

- Closed formula: formula with no free variables.

Example



$$p \longrightarrow \forall q \exists p (q \leftrightarrow p) \vee r$$

Only Free Variables Matter

The truth value of a formula depends only on the truth values of free variables of the formula:

Lemma. Let for all free variables p of F we have $I_1(p) = I_2(p)$. Then $I_1 \models F$ if and only if $I_2 \models F$.

Validity and satisfiability are defined as for propositional formulas.

There is no difference between these notions for closed formulas:

Lemma. For every interpretation I and closed formula F the following propositions are equivalent: (i) $I \models F$; (ii) F is satisfiable; and (iii) F is valid.

Validity and satisfiability can be expressed through truth:

Lemma. Let F be a formula with free variables $p_1, \ldots, p_n$.

- F is satisfiable if and only if the formula $\exists p_1 \ldots \exists p_n F$ is satisfiable (true, valid).
- F is valid if and only if the formula $\forall p_1 \ldots \forall p_n F$ is valid (true, satisfiable).

Equivalent replacement

Lemma Let I be an interpretation and $I \models F_1 \leftrightarrow F_2$. Then $I \models G[F_1] \leftrightarrow G[F_2]$.

Theorem (Equivalent Replacement) Let $F_1 \equiv F_2$. Then Then $G[F_1] \equiv G[F_2]$.

Monotonic replacement

Monotonic Replacement Lemma. Let $I \models F_1 \rightarrow F_2$. If $pol(G, \pi) = 1$, then $I \models G[F_1]_\pi \rightarrow G[F_2]_\pi$. Likewise, if $pol(G, \pi) = -1$, then $I \models G[F_2]_\pi \rightarrow G[F_1]_\pi$.

Monotonic Replacement Theorem. Let $F_1 \rightarrow F_2$ be valid. If $pol(G, \pi) = 1$, then $G[F_1]_\pi \rightarrow G[F_2]_\pi$ is valid too. Likewise, if $pol(G, \pi) = -1$, then $G[F_2]_\pi \rightarrow G[F_1]_\pi$ is valid too.

Substitutions for propositional formulas

Substitution: $(F)_p^G$

Example: $$((p \vee s) \wedge (q \rightarrow p))_p^{(l \wedge s)} =$$
$$(((l \wedge s) \vee s) \wedge (q \rightarrow (l \wedge s)))$$

Properties: If we apply any substitution to a valid formula then we also obtain a valid formula.

Substitution for quantified formulas

Some problems...

Consider $\exists q(\neg p \leftrightarrow q)$.

We cannot simply replace variables by formulas any more:

$\exists(r \rightarrow r)(\neg p \leftrightarrow r \rightarrow r)$???

Free variables are parameters.

We substitute only parameters.

But a variable can have both free and bound occurrences in a formula.

$(\forall p p \rightarrow q) \wedge (q \vee (q \rightarrow p))$

Renaming bound variables

Notation: $\exists\!\!\!\forall$: any of $\forall$, $\exists$.

$\wedge\!\!\!\!\vee$: any of $\wedge$, $\vee$.

Renaming bound variables in F:

Let $F[\,\exists\!\!\!\forall pG\,]$ and q be a variable not occurring in F then we replace all free occurrences of p in G by q obtaining G', and the result of renaming is $F[\,\exists\!\!\!\forall qG'\,]$.

Lemma. $F[\,\exists\!\!\!\forall pG\,] \equiv F[\,\exists\!\!\!\forall qG'\,]$.

Example:

$$\exists q(\forall p((p \to q) \wedge p)) \vee p$$

Then we can rename p by r.

$$\exists q(\forall r((r \to q) \wedge r)) \vee p$$

Rectified formulas

Rectified formula F:

- ▷ no variable appears both free and bound in F;
- ▷ for every variable p, the formula F contains at most one occurrence of quantifiers $\exists\forall p$.

Any formula can be transformed into a rectified formula by renaming bound variables.

Now we can use usual notation $(F)_p^G$ assuming that p occurs only free.

Rectification: Example

$$p \rightarrow \exists p(p \wedge \forall p(p \vee r \rightarrow \neg p)) \Rightarrow$$

$$p \rightarrow \exists p_1(p_1 \wedge \forall p(p \vee r \rightarrow \neg p)) \Rightarrow$$

$$p \rightarrow \exists p_1(p_1 \wedge \forall p_2(p_2 \vee r \rightarrow \neg p_2))$$

Another problem

$\exists q(\neg p \leftrightarrow q)$: there exists a truth value equivalent to $\neg p$.

Replace p by q.

$\exists q(\neg q \leftrightarrow q)$: there exists a truth value equivalent to its own negation.

Another restriction

Let we want to substitute $(F)_p^G$.

Then we require: no free variable in G become bound in $(F)_p^G$.

In previous example:

$\exists q(\neg p \leftrightarrow q)$

Substitute p by q. ($\exists q(\neg q \leftrightarrow q)$ does not satisfy above)

Uniform solution – renaming of bound variables

$\exists q(\neg p \leftrightarrow q) \equiv \exists r(\neg p \leftrightarrow r)$

Now we can substitute p by q: $\exists r(\neg q \leftrightarrow r)$

From now on we always assume that:

(1) formulas are rectified.

(2) all substitutions satisfy the requirement above

Prenex form

- Quantifier-free formula: no quantifiers (that is, propositional).
- Prenex formula has the form $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n G$, where G is quantifier-free.
- Outermost prefix of $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n G$: the longest subsequence $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_k p_k$ of $\exists\!\!\!\forall_1 p_1 \ldots \exists\!\!\!\forall_n p_n$ such that $\exists\!\!\!\forall_1 = \ldots = \exists\!\!\!\forall_k$.
- A formula F is a prenex form of a formula G if F is prenex and $F \equiv G$.

Prenexing rules:

$$\exists\!\!\!\forall p F_1 \wedge\!\!\!\!\vee \ldots \wedge\!\!\!\!\vee F_n \;\Rightarrow\; \exists\!\!\!\forall p (F_1 \wedge\!\!\!\!\vee \ldots \wedge\!\!\!\!\vee F_n)$$

$$\forall p F_1 \rightarrow F_2 \;\Rightarrow\; \exists p (F_1 \rightarrow F_2) \qquad \exists p F_1 \rightarrow F_2 \;\Rightarrow\; \forall p (F_1 \rightarrow F_2)$$

$$F_1 \rightarrow \forall p F_2 \;\Rightarrow\; \forall p (F_1 \rightarrow F_2) \qquad F_1 \rightarrow \exists p F_2 \;\Rightarrow\; \exists p (F_1 \rightarrow F_2)$$

$$\neg \forall p F \;\Rightarrow\; \exists p \neg F \qquad\qquad \neg \exists p F \;\Rightarrow\; \forall p \neg F$$

Prenexing. Example I

$$\exists q (q \rightarrow p) \rightarrow \neg \forall r (r \rightarrow p) \lor p \Rightarrow$$

$$\forall q ((q \rightarrow p) \rightarrow \neg \forall r (r \rightarrow p) \lor p) \Rightarrow$$

$$\forall q ((q \rightarrow p) \rightarrow \exists r \neg (r \rightarrow p) \lor p) \Rightarrow$$

$$\forall q ((q \rightarrow p) \rightarrow \exists r (\neg (r \rightarrow p) \lor p)) \Rightarrow$$

$$\forall q \exists r ((q \rightarrow p) \rightarrow \neg (r \rightarrow p) \lor p).$$

Prenexing. Example II

$$\exists q(q \rightarrow p) \rightarrow \neg\forall r(r \rightarrow p) \vee p \Rightarrow$$

$$\exists q(q \rightarrow p) \rightarrow \exists r\neg(r \rightarrow p) \vee p \Rightarrow$$

$$\exists q(q \rightarrow p) \rightarrow \exists r(\neg(r \rightarrow p) \vee p) \Rightarrow$$

$$\exists r(\exists q(q \rightarrow p) \rightarrow \neg(r \rightarrow p) \vee p) \Rightarrow$$

$$\exists r\forall q((q \rightarrow p) \rightarrow \neg(r \rightarrow p) \vee p).$$

What's next

Algorithms for satisfiability, validity of QBF:

- ▷ Splitting
- ▷ DLL

Reminder:

(i) $F(p_1, \dots, p_n)$ is satisfiable iff $\exists p_1 \dots \exists p_n F(p_1, \dots, p_n)$ is true

(ii) $F(p_1, \dots, p_n)$ is valid iff $\forall p_1 \dots \forall p_n F(p_1, \dots, p_n)$ is true

Algorithms will check truth/falsity of closed formulas.