Propositional Logic of Finite Domains

- Similar to propositional logic, but instead of two-valued boolean variables uses variables ranging over a finite domain of elements.
- Motivation: system modeling.
- Definition and examples.
- Reasoning using a tableau system.
- Propositional logic as an instance of PLFD.
- Translation of PLFD into propositional logic.

State-changing systems

Informally	Formally
At each time moment, the system is in a particular state.	This state can be characterized by values of some variables, called the state variables.
The system state is changing in time. There are actions (controlled or not) that change the state.	Actions change values of some state variables.

Examples

1. Reactive systems. These are the systems that interact with their environment.

2. Concurrent systems. These are the systems which consist of a set of components functioning together. Usually, functioning of these components can be described indepedently, but they communicate through shared variables or some kind of communication channels, for example queues.

Reasoning about state-changing systems

1. Build a formal model of this state-changing system which describes, in particular, functioning of the system, or some abstraction thereof.

2. Using a logic to specify and verify properties of the system.

Microwave Reasoning

variable	domain of values
selected_button	$\{none, micro, grill, defrost\}$
is_on	$\{0, 1\}$
content	$\{none, burger, pizza, cabbage\}$
user	$\{nobody, student, veggie, mcdonald\}$
state	$\{still, cooking, defrost\}$

Propositional Logic of Finite Domains (PLFD)

PLFD is a family of logics. Each instance of PLFD is characterized by

- a set X of variables;
- a mapping dom, such that for every $x \in X$, $dom(x)$ is a non-empty finite set, called the domain for x.

Syntax of PLFD

Formulas

- If x is a variable and $v \in dom(x)$ is a value in the domain of x, then $x = v$ is a formula, also called atomic formula, or simply atom.
- Other formulas are built from atomic formulas as in propositional logic, using the connectives $\top$, $\bot$, $\wedge$, $\vee$, $\neg$, $\rightarrow$, and $\leftrightarrow$.

Semantics

- ▷ Interpretation for a set of variables X is a mapping I from X to the set of values such that for all $x \in X$ we have $I(x) \in dom(x)$.
- ▷ Extend interpretations to mappings from formulas to boolean values.
  1. $I(x = v) = 1$ if and only if $I(x) = v$.
  2. If A is not atomic, then as for propositional formulas.
- ▷ The definitions of truth, models, validity, satisfiability, and equivalence are defined exactly as in propositional logic.

Example

Let a variable x range over the domain $\{a, b, c\}$, that is $dom(x) = \{a, b, c\}$. Then the following formula is valid:

$$\neg x = a \rightarrow x = b \vee x = c.$$

Propositional Logic as PLFD

The domain for each variable is $\{0, 1\}$.

Instead of atoms p use $p = 1$.

In general, when p is a boolean variable, that is, $dom(p) = \{0, 1\}$, in PLFD we will write p instead of $p = 1$.

Translation of PLFD into Propositional Logic

- ▷ Introduce a propositional variable x_v for each variable x and value $v \in dom(x)$.
- ▷ Replace every atom $x = v$ by x_v;
- ▷ Add domain axiom for each variable x:

$$(x_{v_1} \vee \ldots \vee x_{v_n}) \wedge \bigwedge_{i<j} (\neg x_{v_i} \vee \neg x_{v_j}),$$

where $dom(x) = \{v_1, \ldots, v_n\}$.

Example

Let x range over the domain $\{a, b, c\}$. To check satisfiability of the following formula

$$\neg(x = b \lor x = c).$$

we have to check satisfiability of the set of formulas

$$(x_a \lor x_b \lor x_c) \land (\neg x_a \lor \neg x_b) \land (\neg x_a \lor \neg x_c) \land (\neg x_b \lor \neg x_c) \land$$
$$\neg(x_b \lor x_c).$$

- ▷ Use signed formulas.
- ▷ Use new kind of atomic formulas $x \in \{v_1, \ldots, v_n\}$, replace $x = a$ by $x \in \{a\}$.
- ▷ Abbreviations: instead of $(x \in D) = 1$ write $x \in D$, instead of $(x \in D) = 0$ write $x \notin D$.
- ▷ New tableau rules:

$$\begin{aligned} x \notin D \quad &\rightsquigarrow \quad x \in dom(x) \setminus D; \\ x \in D_1,\, x \in D_2 \quad &\rightsquigarrow \quad x \in D_1 \cap D_2. \end{aligned}$$

- ▷ A branch is closed if it contains $x \in \{\}$.

Example, propositional logic

1. war: one can start a war;
2. guilty: the country is guilty;
3. has: the country has weapons of mass destruction.

Example, propositional logic

If a country has weapons of mass destruction, then it is guilty. To start a war against the country one had to have a very good reason, possessing weapon of mass destruction is such a reason. We would like to check whether, under the above assumptions, it is possible that a war started against a country that is not guilty.

$$\mathsf{has} \longrightarrow \mathsf{guilty},$$

$$\mathsf{war} \longrightarrow \mathsf{has},$$

$$\mathsf{war},$$

$$\neg\mathsf{guilty}$$

Add a third value to a variable

Now let us consider a slightly different situation, when the domain of the variable has consists of the values yes, no, and a third value, for example, $suspected$.

Example, propositional logic

If a country has weapons of mass destruction, then it is guilty. To start a war against the country one had to have a very good reason, possessing weapon of mass destruction is such a reason. We would like to check whether, under the above assumptions, it is possible that a war started against a country that is not guilty.

$\text{has} \rightarrow \text{guilty},$

$\text{war} \rightarrow \text{has},$

$\text{war},$

$\neg\text{guilty}$

$\text{has} = yes \rightarrow \text{guilty},$

$\text{war} \rightarrow \neg(\text{has} = no),$

$\text{war},$

$\neg\text{guilty}$

Example

$$(\text{has} \in \{yes\} \rightarrow \text{guilty}) = 1 \qquad \text{(a)}$$
$$(\text{war} \rightarrow \neg(\text{has} \in \{no\})) = 1 \qquad \text{(b)}$$
$$\text{war} = 1$$
$$(\neg\text{guilty}) = 1 \qquad \text{(c)}$$

Example

$(\text{has} \in \{yes\} \rightarrow \text{guilty}) = 1$ (a)

$(\text{war} \rightarrow \neg(\text{has} \in \{no\})) = 1$ (b)

$\text{war} = 1$

$(\neg\text{guilty}) = 1$ (c)

$\Big|$ c

$\text{guilty} = 0$

Example

$(\textsf{has} \in \{yes\} \rightarrow \textsf{guilty}) = 1$ (a)

$(\textsf{war} \rightarrow \neg(\textsf{has} \in \{no\})) = 1$ (b)

$\textsf{war} = 1$

$(\neg\textsf{guilty}) = 1$ (c)

| c

$\textsf{guilty} = 0$

/ a \

(d) $\textsf{has} \notin \{yes\}$ $\textsf{guilty} = 1$

closed

Example

$(\mathsf{has} \in \{yes\} \rightarrow \mathsf{guilty}) = 1$ (a)
$(\mathsf{war} \rightarrow \neg(\mathsf{has} \in \{no\})) = 1$ (b)
$\mathsf{war} = 1$
$(\neg\mathsf{guilty}) = 1$ (c)

| c

$\mathsf{guilty} = 0$

/ a \

(d) $\mathsf{has} \notin \{yes\}$ $\mathsf{guilty} = 1$

| d closed

(e) $\mathsf{has} \in \{no, suspected\}$

Example

$(\text{has} \in \{yes\} \rightarrow \text{guilty}) = 1$ (a)

$(\text{war} \rightarrow \neg(\text{has} \in \{no\})) = 1$ (b)

$\text{war} = 1$

$(\neg\text{guilty}) = 1$ (c)

| c

$\text{guilty} = 0$

/ a \

(d) $\text{has} \notin \{yes\}$ $\quad$ $\text{guilty} = 1$

| d $\quad$ closed

(e) $\text{has} \in \{no, suspected\}$

/ b \

$\text{war} = 0$ $\quad$ $(\neg(\text{has} \in \{no\})) = 1$ (f)

closed

Example

$(\text{has} \in \{yes\} \rightarrow \text{guilty}) = 1$ (a)
$(\text{war} \rightarrow \neg(\text{has} \in \{no\})) = 1$ (b)
$\text{war} = 1$
$(\neg\text{guilty}) = 1$ (c)

| c

$\text{guilty} = 0$

/ a \

(d) $\text{has} \notin \{yes\}$ $\text{guilty} = 1$

| d closed

(e) $\text{has} \in \{no, suspected\}$

/ b \

$\text{war} = 0$ $(\neg(\text{has} \in \{no\})) = 1$ (f)

closed | f

$\text{has} \notin \{no\}$ (g)

Example

$(\text{has} \in \{yes\} \rightarrow \text{guilty}) = 1$ (a)

$(\text{war} \rightarrow \neg(\text{has} \in \{no\})) = 1$ (b)

$\text{war} = 1$

$(\neg\text{guilty}) = 1$ (c)

| c

$\text{guilty} = 0$

/ a \

(d) $\text{has} \notin \{yes\}$ $\text{guilty} = 1$

| d closed

(e) $\text{has} \in \{no, suspected\}$

/ b \

$\text{war} = 0$ $(\neg(\text{has} \in \{no\})) = 1$ (f)

closed | f

$\text{has} \notin \{no\}$ (g)

| g

$\text{has} \in \{yes, suspected\}$ (h)

Example

$(\text{has} \in \{yes\} \to \text{guilty}) = 1$ (a)

$(\text{war} \to \neg(\text{has} \in \{no\})) = 1$ (b)

$\text{war} = 1$

$(\neg\text{guilty}) = 1$ (c)

| c

$\text{guilty} = 0$

/ a \

(d) $\text{has} \notin \{yes\}$ $\text{guilty} = 1$

| d closed

(e) $\text{has} \in \{no, suspected\}$

/ b \

$\text{war} = 0$ $(\neg(\text{has} \in \{no\})) = 1$ (f)

closed | f

$\text{has} \notin \{no\}$ (g)

| g

$\text{has} \in \{yes, suspected\}$ (h)

| e,h

$\text{has} \in \{suspected\}$

Example

$(\text{has} \in \{yes\} \rightarrow \text{guilty}) = 1$ (a)

$(\text{war} \rightarrow \neg(\text{has} \in \{no\})) = 1$ (b)

$\text{war} = 1$

$(\neg\text{guilty}) = 1$ (c)

| c

$\text{guilty} = 0$

/ a \

(d) $\text{has} \notin \{yes\}$ $\qquad$ $\text{guilty} = 1$

| d $\qquad$ closed

(e) $\text{has} \in \{no, suspected\}$

/ b \

$\text{war} = 0$ $\qquad$ $(\neg(\text{has} \in \{no\})) = 1$ (f)

closed $\qquad$ | f

$\text{has} \notin \{no\}$ (g)

| g

$\text{has} \in \{yes, suspected\}$ (h)

| e,h

$\text{has} \in \{suspected\}$