

## *Propositional logic: syntax*

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Assume a countable set of **boolean variables**.

**Propositional formula:**

- ▶ Every boolean variable is a formula, also called **atomic formula**, or simply **atom**.
- ▶  $\top$  and  $\perp$  are formulas.
- ▶ If  $A_1, \dots, A_n$  are formulas, where  $n \geq 2$ , then  $(A_1 \wedge \dots \wedge A_n)$  and  $(A_1 \vee \dots \vee A_n)$  are formulas.
- ▶ If  $A$  is a formula, then  $\neg A$  is a formula.
- ▶ If  $A$  and  $B$  are formulas, then  $(A \rightarrow B)$  and  $(A \leftrightarrow B)$  are formulas.

## Connectives

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Connective	Name	Priority
$\top$	verum	
$\perp$	falsum	
$\neg$	negation	4
$\wedge$	conjunction	3
$\vee$	disjunction	3
$\rightarrow$	implication	2
$\leftrightarrow$	equivalence	1

## *Parsing Formulas*

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As usual, we will sometimes omit parenthesis in formulas and use priorities to disambiguate them. For example,

$$\neg A \wedge B \rightarrow C \vee D \leftrightarrow E$$

is parsed as

$$(((\neg A) \wedge B) \rightarrow (C \vee D)) \leftrightarrow E.$$

## *Semantics, Interpretation*

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- ▶ A **boolean value**, also called a **truth value**, is either **1** or **0**.
- ▶ A (partial) **interpretation** for a set  $P$  of boolean variables: a (partial) mapping  $I : P \rightarrow \{1, 0\}$ .
- ▶ Interpretations are also called **truth assignments**.

## Truth value

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Extend  $I$  to all formulas:

1.  $I(\top) = 1$  and  $I(\perp) = 0$ .
2.  $I(A_1 \wedge \dots \wedge A_n) = 1$  if and only if  $I(A_i) = 1$  for all  $i$ .
3.  $I(A_1 \vee \dots \vee A_n) = 1$  if and only if  $I(A_i) = 1$  for some  $i$ .
4.  $I(\neg A) = 1$  if and only if  $I(A) = 0$ .
5.  $I(A \rightarrow B) = 1$  if and only if  $I(A) = 0$  or  $I(B) = 1$ .
6.  $I(A \leftrightarrow B) = 1$  if and only if  $I(A) = I(B)$ .

## Operation tables

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$\wedge$	1	0	$\vee$	1	0	$\neg$	
1	1	0	1	1	1	1	0
0	0	0	0	1	0	0	1

$\rightarrow$	1	0	$\leftrightarrow$	1	0
1	1	0	1	1	0
0	1	1	0	0	1

## *Satisfiability, validity*

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- ▶ If a formula  $A$  is true in  $I$  we say that  $I$  satisfies  $A$  and that  $I$  is a model of  $A$ , denoted by  $I \models A$ .
- ▶  $A$  is **satisfiable** (**valid**) if it is true in some (every) interpretation.
- ▶ Two formulas  $A$  and  $B$  are called **equivalent**, denoted  $A \equiv B$  if they have the same models.

## *Examples*

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$A \rightarrow A$  and  $A \vee \neg A$  are **valid** for all formulas  $A$ .

Evidently, every **valid** formula is also **satisfiable**.

$A \wedge \neg A$  is **unsatisfiable**.

Formula  $p$  is **satisfiable** but **not valid**.

## *Examples: equivalences*

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For all formulas  $A$  and  $B$ , the following equivalences hold.

$$A \rightarrow \perp \equiv \neg A; \quad (1)$$

$$\top \rightarrow A \equiv A; \quad (2)$$

$$A \rightarrow B \equiv \neg(A \wedge \neg B); \quad (3)$$

$$A \wedge B \equiv \neg(\neg A \vee \neg B); \quad (4)$$

$$A \vee B \equiv \neg A \rightarrow B. \quad (5)$$

## *Connections between these notions*

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1. A formula  $A$  is valid if and only if  $\neg A$  is unsatisfiable.
2. A formula  $A$  is satisfiable if and only if  $\neg A$  is not valid.
3. A formula  $A$  is valid if and only if  $A$  is equivalent to  $\top$ .
4. Formulas  $A$  and  $B$  are equivalent if and only if the formula  $A \leftrightarrow B$  is valid.

## *Notation*

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We denote by  $A[B]$  a formula  $A$  with a fixed occurrence of a subformula  $B$ . If we use this notation we can also write  $A[B']$  to denote the formula obtained from  $A$  by replacing this occurrence of  $B$  by  $B'$ .

## *Equivalent replacement*

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**Lemma** Let  $I$  be an interpretation and  $I \models A_1 \leftrightarrow A_2$ . Then  $I \models B[A_1] \leftrightarrow B[A_2]$ .

**Theorem (Equivalent Replacement)** Let  $A_1 \equiv A_2$ . Then  $B[A_1] \equiv B[A_2]$ .

## *How to evaluate a formula?*

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Let's evaluate the formula

$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$$

in the interpretation

$$\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}.$$

## Evaluating a formula. 1

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	formula	value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	

## Evaluating a formula. 2

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	formula	value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	
2		$p \rightarrow r$
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	

## Evaluating a formula. 3

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	formula	value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	
2		$p \rightarrow r$
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	
4		$p \wedge q \rightarrow r$
5	$p \rightarrow q$	

## Evaluating a formula. 4

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	formula	value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	
2		$p \rightarrow r$
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	
4	$p \wedge q \rightarrow r$	
5	$p \rightarrow q$	
6	$p \wedge q$	
7		
9	$r$	

## Evaluating a formula. 5

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	formula	value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	
2	$p \rightarrow r$	
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	
4	$p \wedge q \rightarrow r$	
5	$p \rightarrow q$	
6	$p \wedge q$	
7	$p$	
9	$r$	$r$

## Evaluating a formula. 6

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	formula			value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$			
2			$p \rightarrow r$	
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$			
4		$p \wedge q \rightarrow r$		
5	$p \rightarrow q$			
6		$p \wedge q$		
7	$p$		$p$	
8		$q$		
9			$r$	$r$

## Evaluating a formula. 7

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	formula				value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$				
2				$p \rightarrow r$	
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$				
4			$p \wedge q \rightarrow r$		
5	$p \rightarrow q$				
6		$p \wedge q$			
7	$p$	$p$		$p$	
8		$q$	$q$		
9			$r$	$r$	

## Evaluating a formula. 8

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	formula				value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$				
2				$p \rightarrow r$	
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$				
4			$p \wedge q \rightarrow r$		
5	$p \rightarrow q$				
6		$p \wedge q$			
7	$p$	$p$		$p$	1
8		$q$	$q$		0
9			$r$	$r$	1

## Evaluating a formula. 9

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	formula				value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$				
2				$p \rightarrow r$	
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$				
4			$p \wedge q \rightarrow r$		
5	$p \rightarrow q$				
6			$p \wedge q$		0
7	$p$	$p$	$p$	$p$	1
8	$q$	$q$			0
9			$r$	$r$	1

## Evaluating a formula. 10

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	formula				value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$				
2				$p \rightarrow r$	
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$				
4			$p \wedge q \rightarrow r$		
5	$p \rightarrow q$				0
6		$p \wedge q$			0
7	$p$	$p$		$p$	1
8		$q$	$q$		0
9			$r$	$r$	1

## Evaluating a formula. 11

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	formula				value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$				
2				$p \rightarrow r$	
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$				
4			$p \wedge q \rightarrow r$		1
5	$p \rightarrow q$				0
6		$p \wedge q$			0
7	$p$	$p$		$p$	1
8		$q$	$q$		0
9			$r$	$r$	1

## Evaluating a formula. 12

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	formula				value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$				
2				$p \rightarrow r$	
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$				0
4			$p \wedge q \rightarrow r$		1
5	$p \rightarrow q$				0
6		$p \wedge q$			0
7	$p$	$p$		$p$	1
8		$q$	$q$		0
9			$r$	$r$	1

## Evaluating a formula. 13

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	formula				value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$				
2				$p \rightarrow r$	1
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$				0
4			$p \wedge q \rightarrow r$		1
5	$p \rightarrow q$				0
6		$p \wedge q$			0
7	$p$	$p$	$p$	$p$	1
8		$q$	$q$		0
9			$r$	$r$	1

## Evaluating a formula. 14

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	formula				value
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$				1
2				$p \rightarrow r$	1
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$				0
4				$p \wedge q \rightarrow r$	1
5	$p \rightarrow q$				0
6				$p \wedge q$	0
7	$p$	$p$	$p$	$p$	1
8	$q$	$q$			0
9			$r$	$r$	1