Position and polarity

- ▷ Position is any sequence of positive integers $a_1, \ldots, a_n$, where $n \geq 0$, written as $a_1.a_2.\,\cdots\,.a_n$.
- ▷ Empty position, denoted by ε: when $n = 0$.
- ▷ Polarity: one of the values $-1, 0, 1$.
- ▷ Position in a formula, subformula at position. Notation: $A|_\pi$.
- ▷ Polarity of subformula at a position. Notation: $pol(A, \pi)$.

Position and Polarity

1. For every formula A, ε is a position in A, $A|_\varepsilon \stackrel{\text{def}}{=} A$ and $pol(A,\varepsilon) \stackrel{\text{def}}{=} 1$;
2. Let $A|_\pi = B$.
   (a) If B has the form $B_1 \wedge \ldots \wedge B_n$ or $B_1 \vee \ldots \vee B_n$, then for all $i \in \{1,\ldots,n\}$ the position $\pi.i$ is a position in A, $A|_{\pi.i} \stackrel{\text{def}}{=} B_i$, and $pol(A,\pi.i) \stackrel{\text{def}}{=} pol(A,\pi)$.
   (b) If B has the form $\neg B_1$, then $\pi.1$ is a position in A, $A|_{\pi.1} \stackrel{\text{def}}{=} B_1$ and $pol(A,\pi.1) \stackrel{\text{def}}{=} -pol(A,\pi)$.
   (c) If B has the form $B_1 \rightarrow B_2$, then $\pi.1$ and $\pi.2$ are positions in A and we have $A|_{\pi.1} \stackrel{\text{def}}{=} B_1$, $A|_{\pi.2} \stackrel{\text{def}}{=} B_2$, $pol(A,\pi.1) \stackrel{\text{def}}{=} -pol(A,\pi)$, $pol(A,\pi.2) \stackrel{\text{def}}{=} pol(A,\pi)$.

(d) If B has the form $B_1 \leftrightarrow B_2$, then for all $i \in \{1, 2\}$ the position $\pi.i$ is a position in A, $A|_{\pi.i} \stackrel{\text{def}}{=} B_i$ and $pol(A, \pi.i) \stackrel{\text{def}}{=} 0$.

Positive and negative occurrences

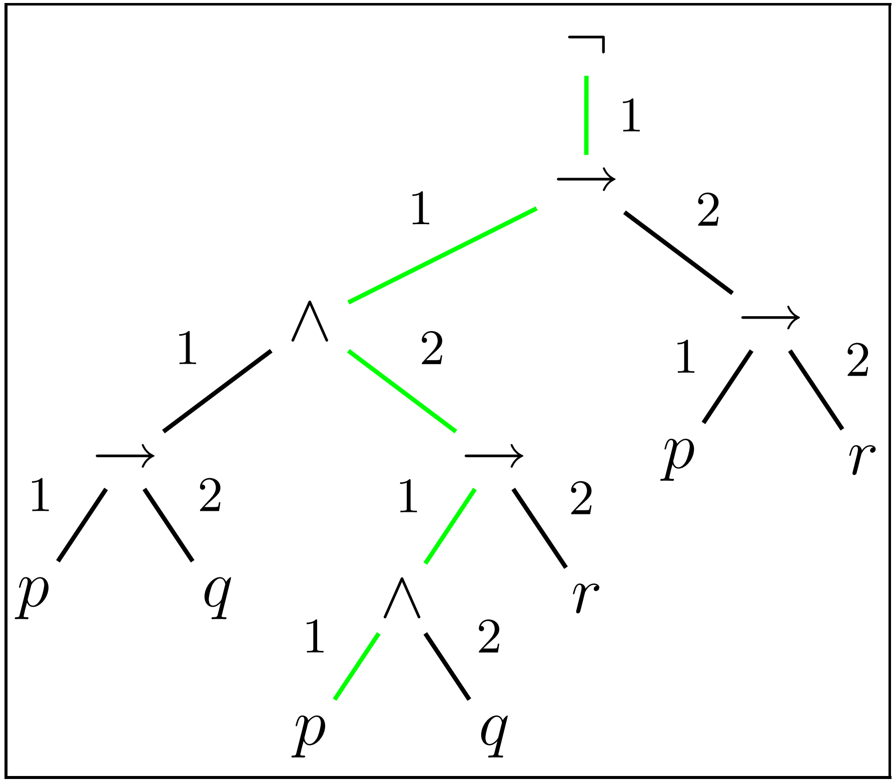
If $A|_\pi = B$, we also say that B occurs in A at the position π.

If, in addition, $pol(A, \pi) = 1$ (respectively, $pol(A, \pi) = -1$), then we call the occurrence of B at the position π in A a positive occurrence (respectively, negative occurrence) of B in A.

Position

Consider the formula

$A \stackrel{\text{def}}{=} \neg((p \to q) \land (p \land q \to r) \to (p \to r))$.



$A|_{1.1.2.1.1} = p$

$A|_{1.1.2.1} = p \land q$

$A|_{1.1.2} = p \land q \to r$

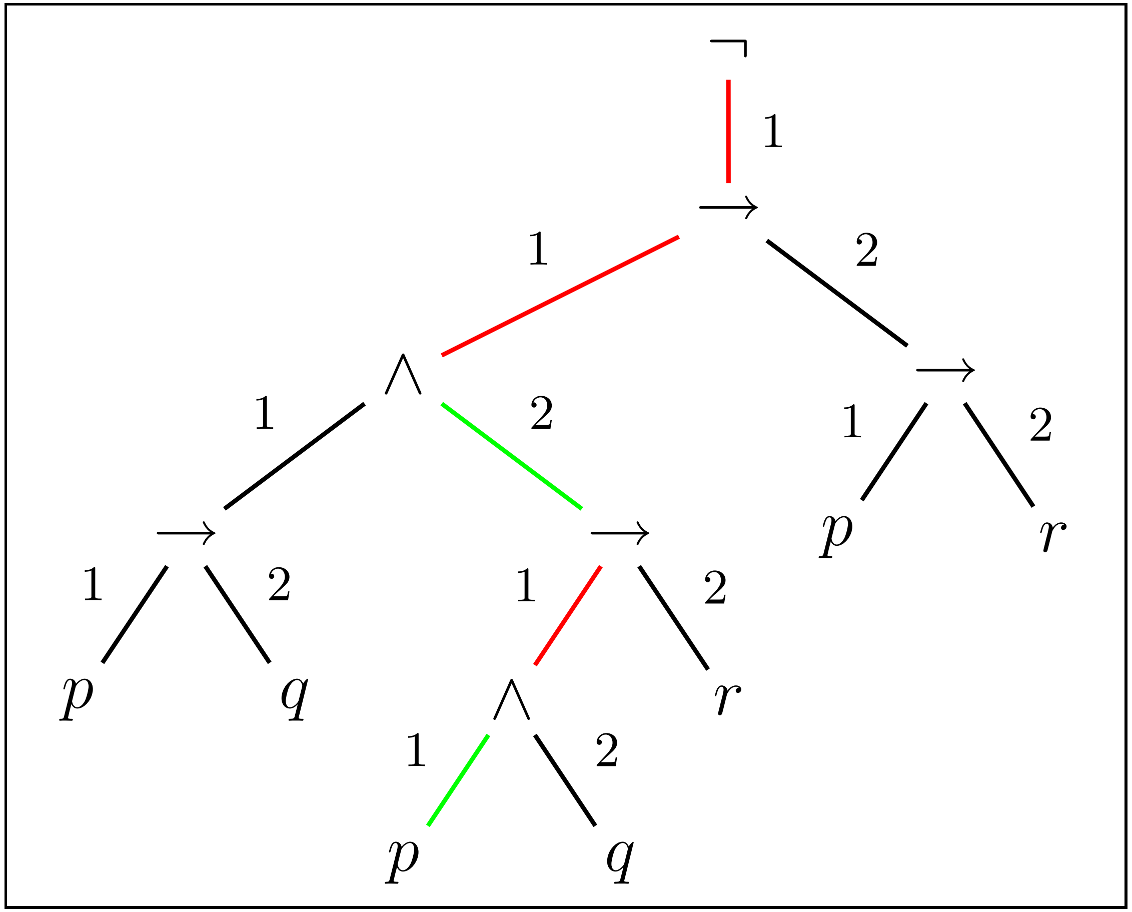
$A|_{1.1} = (p \to q) \land (p \land q \to r)$

$A|_{1} = (p \to q) \land (p \land q \to r) \to (p \to r)$

$A|_{\varepsilon} = A$

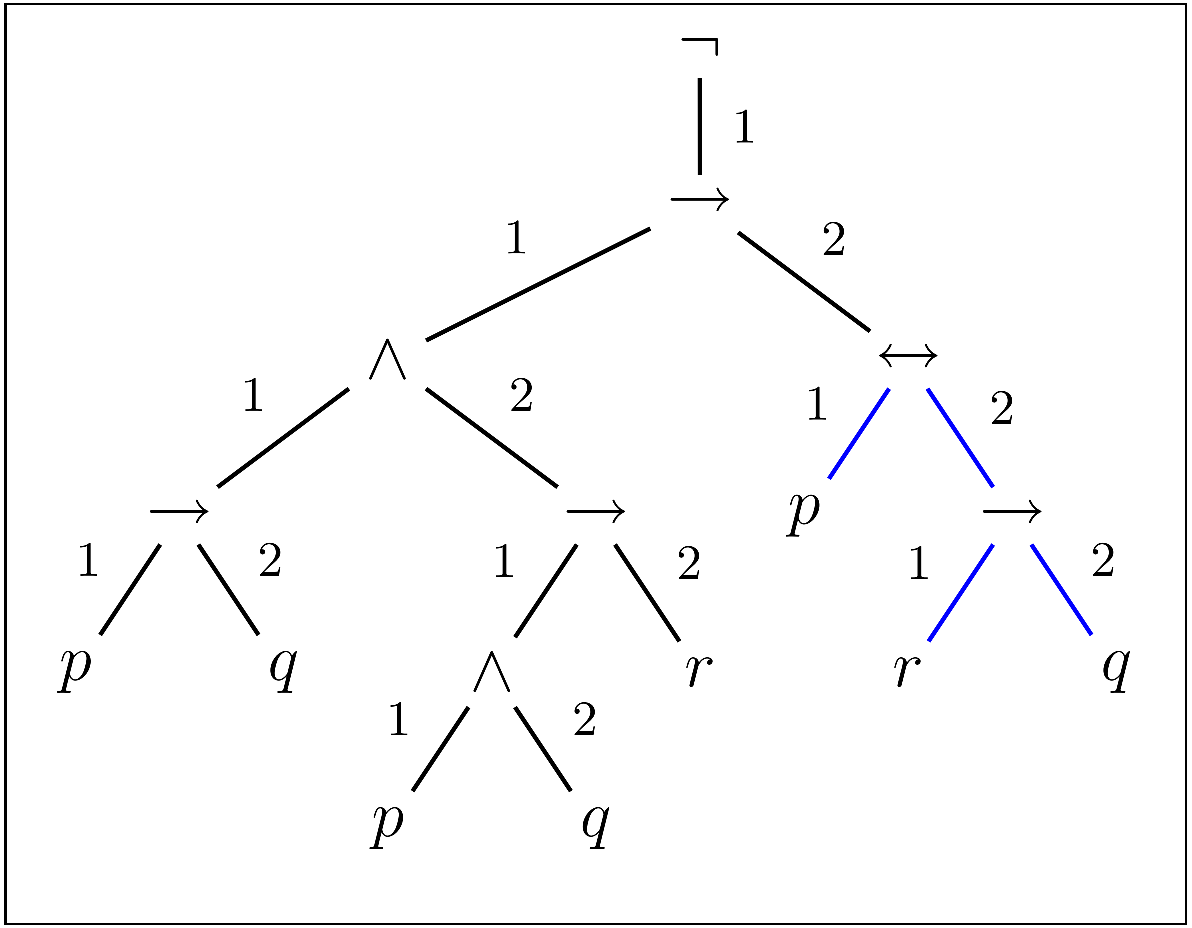
Position, polarity

Consider the formula $\neg((p \to q) \land (p \land q \to r) \to (p \to r))$.



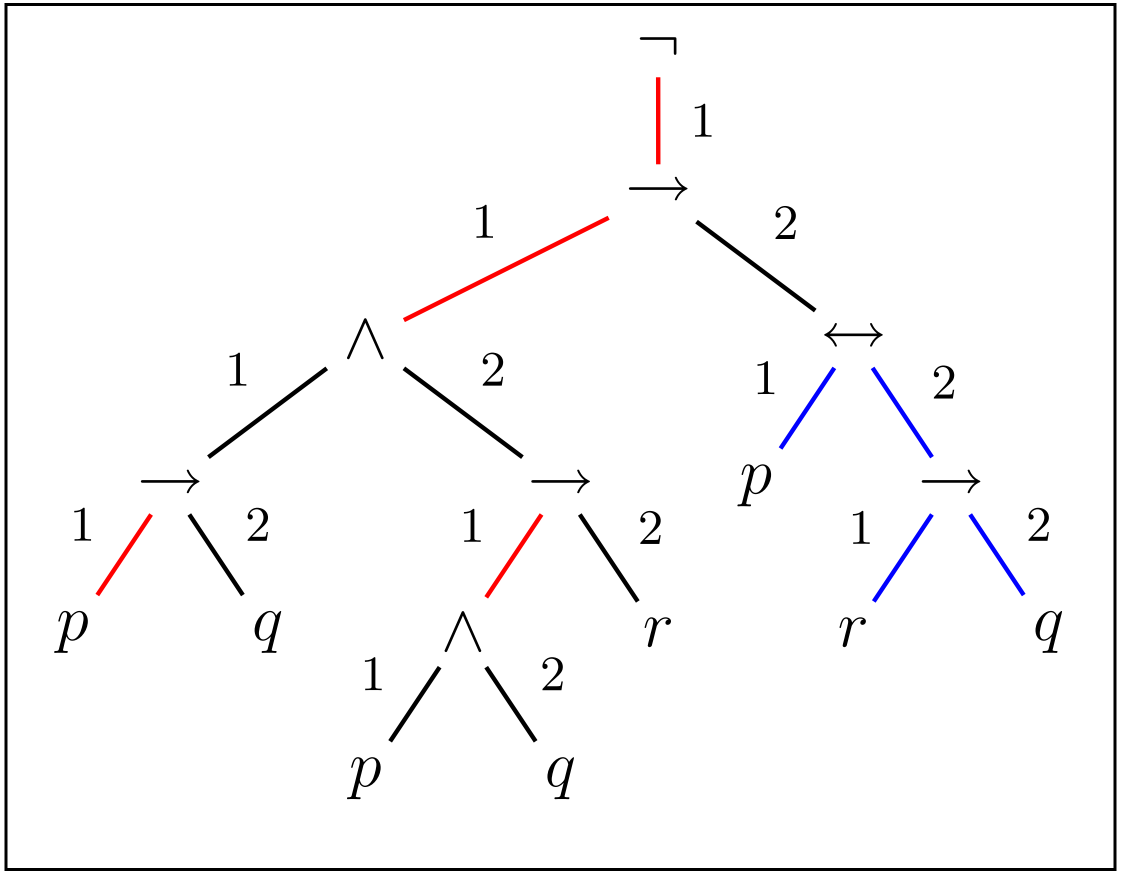
The coloring algorithm for determining polarity

Consider the formula $\neg((p \to q) \land (p \land q \to r) \to (p \leftrightarrow (r \to q)))$. Color in blue all arcs below an equivalence:



The coloring algorithm for determining polarity

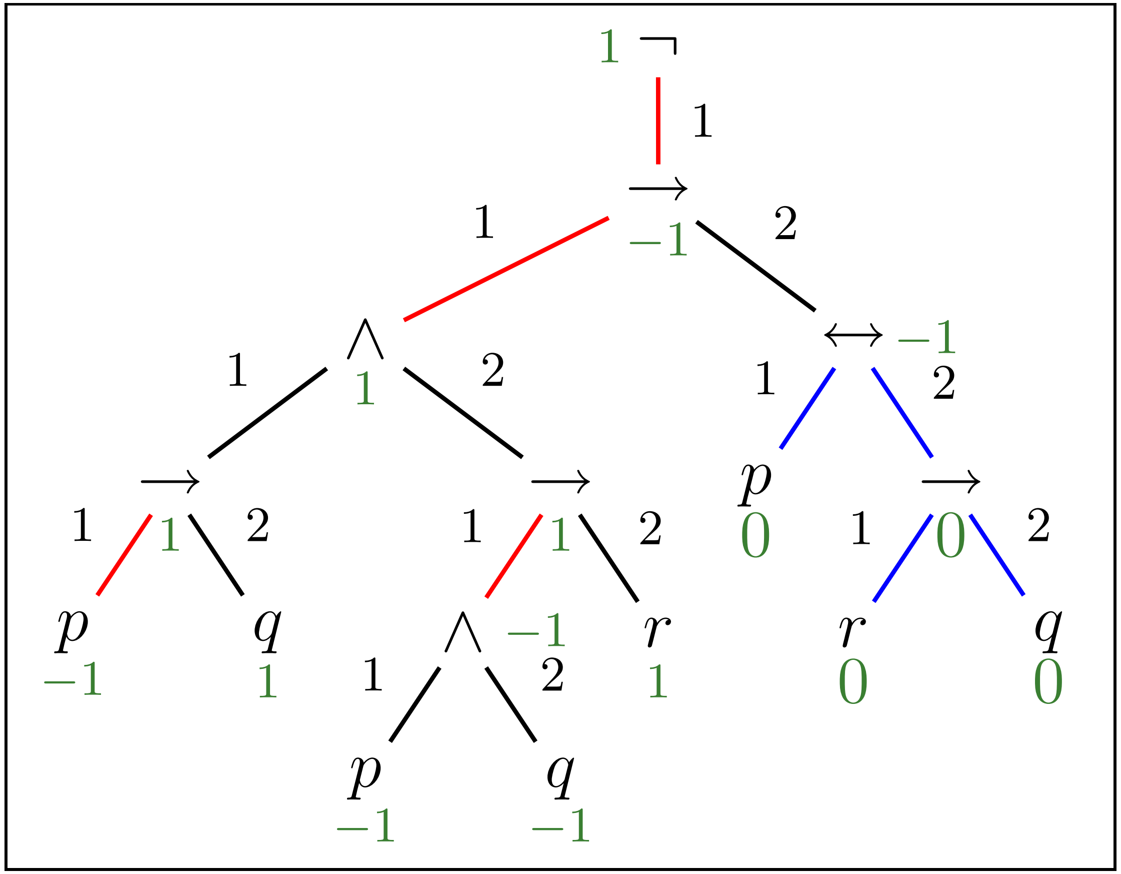
Color in red all arcs going down from a negation or left-hand side of an implication.



The coloring algorithm for determining polarity

If a position has at least one blue arc above it, its polarity is 0.

Otherwise, its polarity is 1 if it has an even number of red arcs above it.



Position and polarity, again

position	subformula	polarity
ε	$\neg((p \to q) \land (p \land q \to r) \to (p \to r))$	1
1	$(p \to q) \land (p \land q \to r) \to (p \to r)$	-1
1.1	$(p \to q) \land (p \land q \to r)$	1
1.1.1	$p \to q$	1
1.1.1.1	p	-1
1.1.1.2	q	1
1.1.2	$p \land q \to r$	1
1.1.2.1	$p \land q$	-1
1.1.2.1.1	p	-1
1.1.2.1.2	q	-1
1.1.2.2	r	1
1.2	$p \to r$	-1
1.2.1	p	1
1.2.2	r	-1

Monotonic replacement

Notation: $A[B]_\pi$:

- ▷ formula A with the subformula B at the position π;
- ▷ formula A with the subformula at the position π replaced by B.

Monotonic Replacement Lemma. Let A, B, B' be formulas, I be an interpretation, and $I \models B \rightarrow B'$. If $pol(A, \pi) = 1$, then $I \models A[B]_\pi \rightarrow A[B']_\pi$. Likewise, if $pol(A, \pi) = -1$, then $I \models A[B']_\pi \rightarrow A[B]_\pi$.

Pure Atom

Atom p is pure in a formula A, if either all occurrences of p in A are positive or all occurrences of p in A are negative.

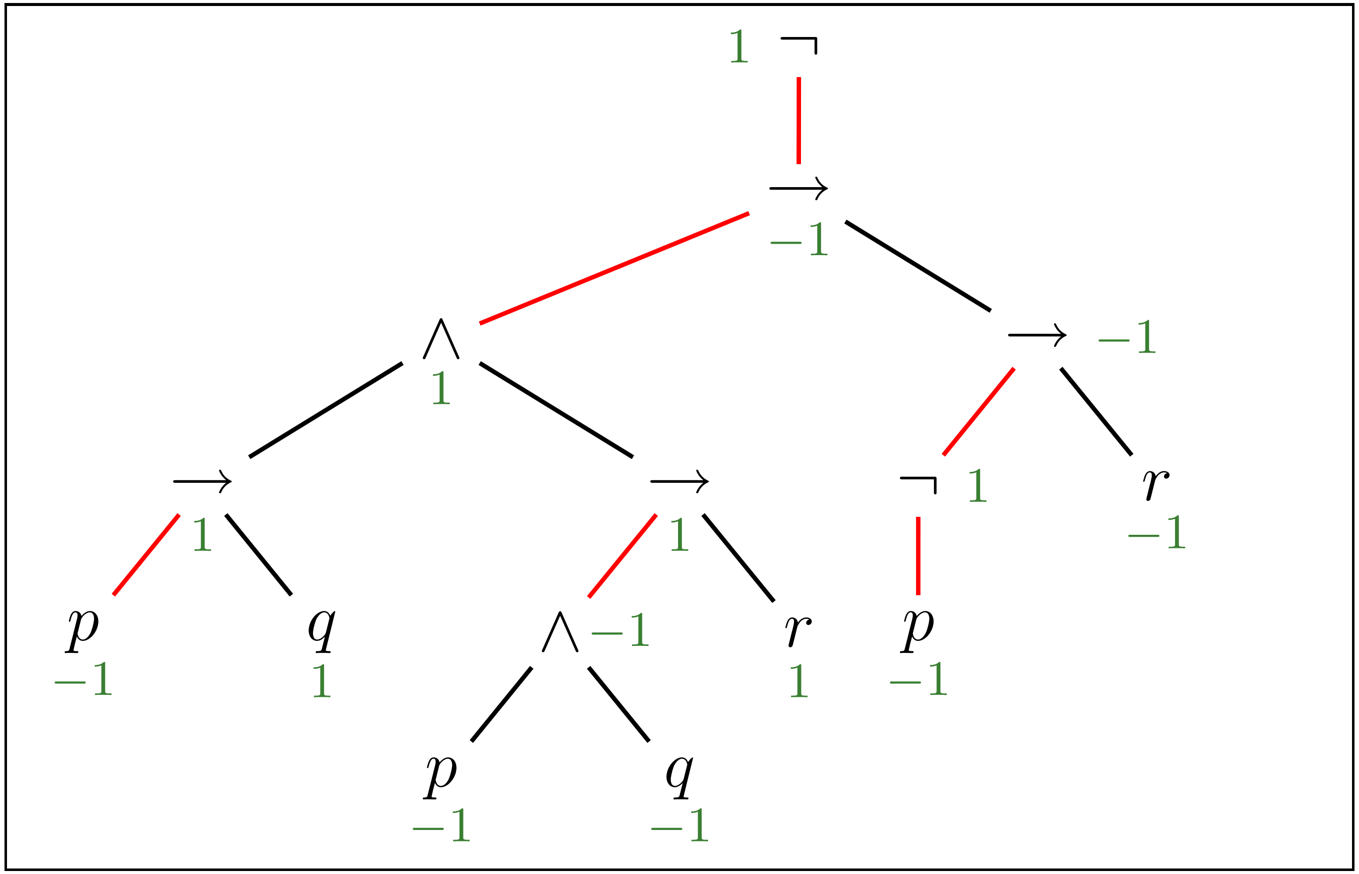
Pure Atom Lemma. Let p be pure in A. Let $I \models A$. Consider I'.

$$I'(q) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } p = q \text{ and } p \text{ occurs in } A \text{ only positively;} \\ 0, & \text{if } p = q \text{ and } p \text{ occurs in } A \text{ only negatively;} \\ I(q), & \text{otherwise.} \end{cases}$$

Then $I' \models A$.

Pure Atom Theorem. Let an atom p has only positive (respectively, only negative) occurrences in A. Then A is satisfiable if and only if so is $A_p^\top$ (respectively, $A_p^\perp$).

Consider $\neg((p \to q) \wedge (p \wedge q \to r) \to (\neg p \to r))$:



Example, continued

$$\neg((p \to q) \land (p \land q \to r) \to (\neg p \to r)) \quad \Rightarrow$$

$$\neg((\bot \to q) \land (\bot \land q \to r) \to (\neg\bot \to r)) \quad \Rightarrow$$

$$\neg(\top \land (\bot \land q \to r) \to (\neg\bot \to r)) \quad \Rightarrow$$

$$\neg((\bot \land q \to r) \to (\neg\bot \to r)) \quad \Rightarrow$$

$$\neg((\bot \to r) \to (\neg\bot \to r)) \quad \Rightarrow$$

$$\neg(\top \to (\neg\bot \to r)) \quad \Rightarrow$$

$$\neg(\neg\bot \to r) \quad \Rightarrow$$

$$\neg(\top \to r) \quad \Rightarrow$$

$$\neg r \quad \Rightarrow$$

$$\neg\bot \quad \Rightarrow$$

$$\top$$

Literal, clause

- ▷ Literal: either an atom p (positive literal) or its negation $\neg p$ (negative literal).
- ▷ The complementary literal to L:

$$\tilde{L} \stackrel{\text{def}}{=} \begin{cases} \neg L, & \text{if } L \text{ is positive;} \\ p, & \text{if } L \text{ has the form } \neg p. \end{cases}$$

  In other words, p and $\neg p$ are complementary.
- ▷ Clause: a disjunction $L_1 \vee \ldots \vee L_n$, $n \geq 0$ of literals.
- ▷ Empty clause, denoted by $\Box$: $n = 0$ (the empty clause is false in every interpretation).
- ▷ Unit clause: $n = 1$.

- A formula A is in conjunctive normal form, or simply CNF, if it is either $\top$, or $\bot$, or a conjunction of disjunctions of literals:

  $$A = \bigwedge_i \bigvee_j L_{i,j}.$$

  (That is, a conjunction of clauses.)

- A formula B is called a conjunctive normal form of a formula A if B is equivalent to A and B is in conjunctive normal form.

Exercises

Exercises: 4.2 (second formula only), 4.4, 4.9 (second formula only), 4.11, 4.16.

Deadline: Friday, February 22nd.