

Satisfiability on CNF

- ▶ An interpretation I satisfies a formula in CNF

$$A = \bigwedge_i \bigvee_j L_{i,j}.$$

if and only if it satisfies every clause

$$\bigvee_j L_{i,j}.$$

in it.

- ▶ An interpretation I satisfies a clause

$$L_1 \vee \dots \vee L_k$$

if and only if it satisfies at least one literal L_m in this clause.

CNF transformation

$$A \leftrightarrow B \Rightarrow (\neg A \vee B) \wedge (\neg B \vee A),$$

$$A \rightarrow B \Rightarrow \neg A \vee B,$$

$$\neg(A \wedge B) \Rightarrow \neg A \vee \neg B,$$

$$\neg(A \vee B) \Rightarrow \neg A \wedge \neg B,$$

$$\neg\neg A \Rightarrow A,$$

$$\begin{aligned} (A_1 \wedge \dots \wedge A_m) \vee B_1 \vee \dots \vee B_n &\Rightarrow (A_1 \vee B_1 \vee \dots \vee B_n) \quad \wedge \\ &\quad \dots \quad \wedge \\ &\quad (A_m \vee B_1 \vee \dots \vee B_n). \end{aligned}$$

CNF, example

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow$$

$$\neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow$$

$$\neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow$$

$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow$$

$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow$$

$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge r \Rightarrow$$

$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow$$

$$(p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow$$

$$(p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

CNF, example

Therefore, the formula

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$$

has the same models as the set consisting of four clauses

$$\neg p \vee q$$

$$\neg p \vee \neg q \vee r$$

$$p$$

$$\neg r$$

The CNF transformation allows one to **reduce the satisfiability problem for formulas to the satisfiability problem for sets of clauses.**

Problem

Compute the CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))).$$

The first step yields:

$$\begin{aligned} p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) &\Rightarrow \\ (\neg p_1 \vee (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) & \\ \wedge (\neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \vee p_1). & \end{aligned}$$

If we continue, the formula will **grow exponentially**.

Idea

Replace subformulas

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))$$

by names:

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow n)))$$

$$n \leftrightarrow (p_5 \leftrightarrow p_6)$$

After several steps

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))$$

$$p_1 \leftrightarrow (p_2 \leftrightarrow n_3);$$

$$n_3 \leftrightarrow (p_3 \leftrightarrow n_4);$$

$$n_4 \leftrightarrow (p_4 \leftrightarrow n_5);$$

$$n_5 \leftrightarrow (p_5 \leftrightarrow p_6).$$

Naming

Lemma. Suppose that S is a set of formulas and B is a formula. Let n be a boolean variable not occurring in $S \cup \{B\}$. Then S is satisfiable if and only if so is the set of formulas $S \cup \{n \leftrightarrow B\}$.

Clausal Form

- ▶ **Clausal form of a formula A :** a set of clauses which is satisfiable if and only if A is satisfiable.
- ▶ **Clausal form of a set S of formulas:** a set of clauses which is satisfiable if and only if so is S .

Definitional Clause Form Transformation

This algorithm converts a formula A into a set of clauses S such that S is a **clausal normal form** of A .

If A has the form $C_1 \wedge \dots \wedge C_n$, where $n \geq 1$ and each C_i is a clause, then $S \stackrel{\text{def}}{=} \{C_1, \dots, C_n\}$.

Otherwise, introduce a name for each subformula B of A such that B is not a literal and use this name instead of the formula.

Example

name	subformula	formula to be transformed to CNF	clauses
			p_1
p_1	$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$	$p_1 \leftrightarrow \neg p_2$	$\neg p_1 \vee \neg p_2$ $p_1 \vee p_2$
p_2	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	$p_2 \leftrightarrow (p_3 \rightarrow p_7)$	$\neg p_2 \vee \neg p_3 \vee p_7$ $p_3 \vee p_2$ $\neg p_7 \vee p_2$
p_3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	$p_3 \leftrightarrow (p_4 \wedge p_5)$	$\neg p_3 \vee p_4$ $\neg p_3 \vee p_5$ $\neg p_4 \vee \neg p_5 \vee p_3$
p_4	$p \rightarrow q$	$p_4 \leftrightarrow (p \rightarrow q)$	$\neg p_4 \vee \neg p \vee q$ $p \vee p_4$ $\neg q \vee p_4$

Example

name	subformula	formula to be transformed to CNF	clauses
p_5	$p \wedge q \rightarrow r$	$p_5 \leftrightarrow (p_6 \rightarrow r)$	$\neg p_5 \vee \neg p_6 \vee r$ $p_6 \vee p_5$ $\neg r \vee p_5$
p_6	$p \wedge q$	$p_6 \leftrightarrow (p \wedge q)$	$\neg p_6 \vee p$ $\neg p_6 \vee q$ $\neg p \vee \neg q \vee p_6$
p_7	$p \rightarrow r$	$p_7 \leftrightarrow (p \rightarrow r)$	$\neg p_7 \vee \neg p \vee r$ $p \vee p_7$ $\neg r \vee p_7$

Searching for satisfiability

$$\neg p \vee q$$

$$\neg p \vee \neg q \vee r$$

$$p$$

$$\neg r$$

Searching for satisfiability

$$\neg p \vee q$$

$$\neg p \vee \neg q \vee r$$

$$p$$

$$\neg r$$

$$q$$

$$\neg q \vee r$$

$$\neg r$$

Searching for satisfiability

$$\begin{array}{l|l} q & q \\ \neg q \vee r & \neg q \\ \neg r & \end{array}$$

Unit propagation

Let S be a set of clauses. **Unit propagation**: repeatedly performing the following transformation: if S contains a unit clause, i.e. a clause consisting of one literal L , then

1. remove from S every clause of the form $L \vee C'$;
2. replace in S every clause of the form $\tilde{L} \vee C'$ by the clause C' .

Searching for satisfiability

p_1	$\neg q \vee p_4$		$\neg q \vee p_4$
$\neg p_1 \vee \neg p_2$	$\neg p_5 \vee \neg p_6 \vee r$	$\neg p_2$	$\neg p_5 \vee \neg p_6 \vee r$
$p_1 \vee p_2$	$p_6 \vee p_5$		$p_6 \vee p_5$
$\neg p_2 \vee \neg p_3 \vee p_7$	$\neg r \vee p_5$	$\neg p_2 \vee \neg p_3 \vee p_7$	$\neg r \vee p_5$
$p_3 \vee p_2$	$\neg p_6 \vee p$	$p_3 \vee p_2$	$\neg p_6 \vee p$
$\neg p_7 \vee p_2$	$\neg p_6 \vee q$	$\neg p_7 \vee p_2$	$\neg p_6 \vee q$
$\neg p_3 \vee p_4$	$\neg p \vee \neg q \vee p_6$	$\neg p_3 \vee p_4$	$\neg p \vee \neg q \vee p_6$
$\neg p_3 \vee p_5$	$\neg p_7 \vee \neg p \vee r$	$\neg p_3 \vee p_5$	$\neg p_7 \vee \neg p \vee r$
$\neg p_4 \vee \neg p_5 \vee p_3$	$p \vee p_7$	$\neg p_4 \vee \neg p_5 \vee p_3$	$p \vee p_7$
$\neg p_4 \vee \neg p \vee q$	$\neg r \vee p_7$	$\neg p_4 \vee \neg p \vee q$	$\neg r \vee p_7$
$p \vee p_4$		$p \vee p_4$	

Unit Propagation

$$\neg p_2$$

$$\neg p_2 \vee \neg p_3 \vee p_7$$

$$p_3 \vee p_2$$

$$\neg p_7 \vee p_2$$

$$\neg p_3 \vee p_4$$

$$\neg p_3 \vee p_5$$

$$\neg p_4 \vee \neg p_5 \vee p_3$$

$$\neg p_4 \vee \neg p \vee q$$

$$p \vee p_4$$

$$\neg q \vee p_4$$

$$\neg p_5 \vee \neg p_6 \vee r$$

$$p_6 \vee p_5$$

$$\neg r \vee p_5$$

$$\neg p_6 \vee p$$

$$\neg p_6 \vee q$$

$$\neg p \vee \neg q \vee p_6$$

$$\neg p_7 \vee \neg p \vee r$$

$$p \vee p_7$$

$$\neg r \vee p_7$$

$$p_3$$

$$\neg p_7$$

$$\neg p_3 \vee p_4$$

$$\neg p_3 \vee p_5$$

$$\neg p_4 \vee \neg p_5 \vee p_3$$

$$\neg p_4 \vee \neg p \vee q$$

$$p \vee p_4$$

$$\neg q \vee p_4$$

$$\neg p_5 \vee \neg p_6 \vee r$$

$$p_6 \vee p_5$$

$$\neg r \vee p_5$$

$$\neg p_6 \vee p$$

$$\neg p_6 \vee q$$

$$\neg p \vee \neg q \vee p_6$$

$$\neg p_7 \vee \neg p \vee r$$

$$p \vee p_7$$

$$\neg r \vee p_7$$

Unit Propagation

p_3	$\neg q \vee p_4$		$\neg q \vee p_4$
$\neg p_7$	$\neg p_5 \vee \neg p_6 \vee r$		$\neg p_5 \vee \neg p_6 \vee r$
$\neg p_3 \vee p_4$	$p_6 \vee p_5$	p_4	$p_6 \vee p_5$
$\neg p_3 \vee p_5$	$\neg r \vee p_5$	p_5	$\neg r \vee p_5$
$\neg p_4 \vee \neg p_5 \vee p_3$	$\neg p_6 \vee p$		$\neg p_6 \vee p$
$\neg p_4 \vee \neg p \vee q$	$\neg p_6 \vee q$	$\neg p_4 \vee \neg p \vee q$	$\neg p_6 \vee q$
$p \vee p_4$	$\neg p \vee \neg q \vee p_6$	$p \vee p_4$	$\neg p \vee \neg q \vee p_6$
	$\neg p_7 \vee \neg p \vee r$		
	$p \vee p_7$		p
	$\neg r \vee p_7$		$\neg r$

Unit Propagation

p_4	$\neg q \vee p_4$
p_5	$\neg p_5 \vee \neg p_6 \vee r$
$\neg p_4 \vee \neg p \vee q$	$p_6 \vee p_5$
$p \vee p_4$	$\neg r \vee p_5$
	$\neg p_6 \vee p$
	$\neg p_6 \vee q$
	$\neg p \vee \neg q \vee p_6$
	p
	$\neg r$

q

$\neg p_6$

$\neg p_6 \vee q$

$\neg q \vee p_6$

Unit Propagation

q

$\neg p_6$

$\neg p_6 \vee q$

$\neg q \vee p_6$

□