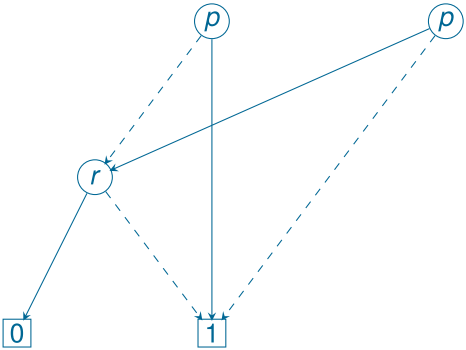
Building OBDDs

procedure *obdd*(F)
input: propositional formula F
parameters: global dag D
output: a node n in (modified) D which represents F
begin
  $F := simplify(F)$
  if $F = \bot$ **then return** $\boxed{0}$
  if $F = \top$ **then return** $\boxed{1}$
  $p := max\_atom(F)$
  $n_1 := obdd(F_p^{\bot})$
  $n_2 := obdd(F_p^{\top})$
  return $integrate(n_1, p, n_2, D)$
end

Example: Building OBDD

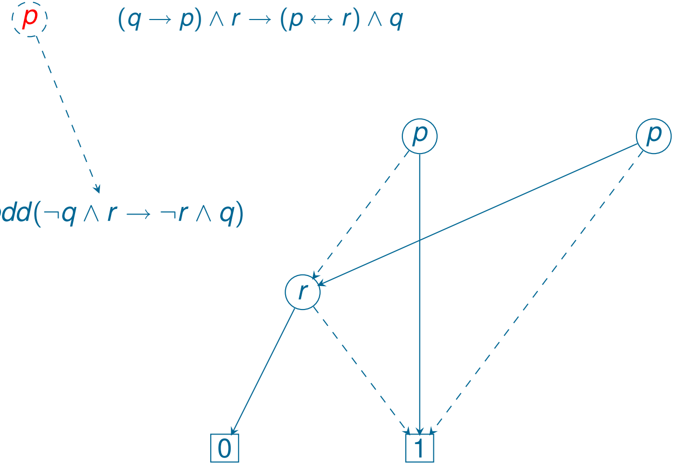
$$obdd((q \to p) \land r \to (p \leftrightarrow r) \land q)$$



Example: Building OBDD

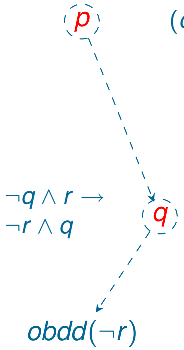
$$(q \to p) \land r \to (p \leftrightarrow r) \land q$$

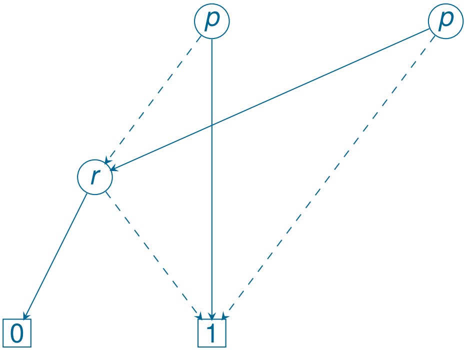
$$obdd(\neg q \land r \to \neg r \land q)$$



Example: Building OBDD

$(q \to p) \land r \to (p \leftrightarrow r) \land q$





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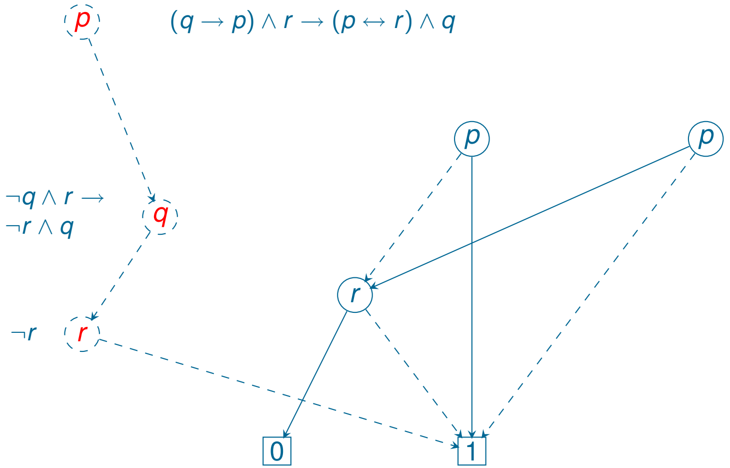

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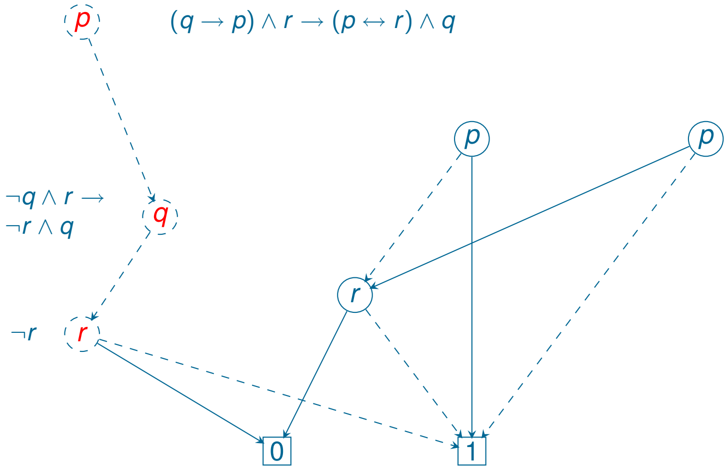
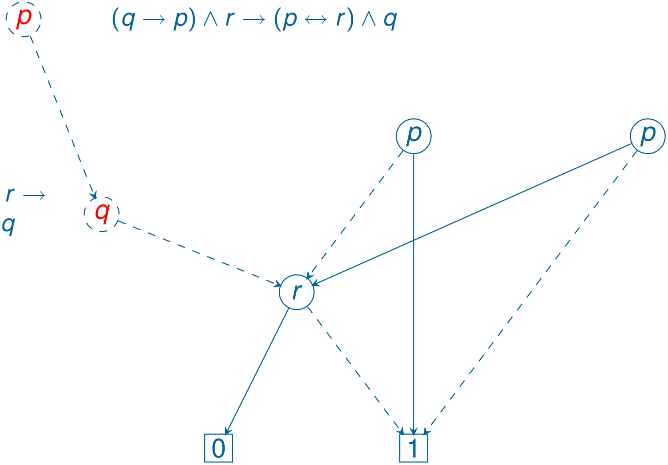


Example: Building OBDD

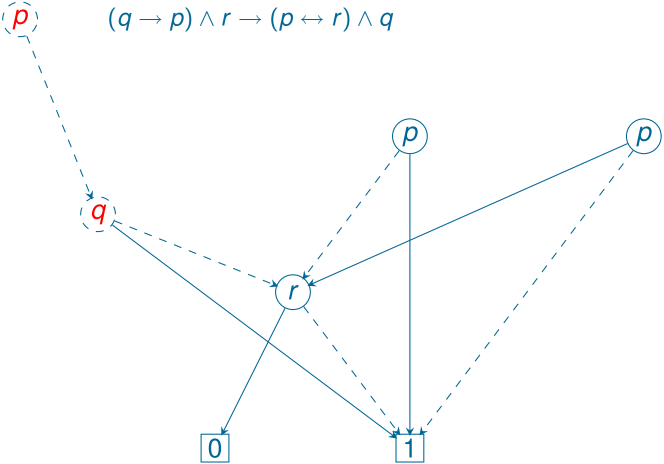
$(q \to p) \land r \to (p \leftrightarrow r) \land q$

$\neg q \land r \to$
$\neg r \land q$



Example: Building OBDD

$(q \rightarrow p) \wedge r \rightarrow (p \leftrightarrow r) \wedge q$



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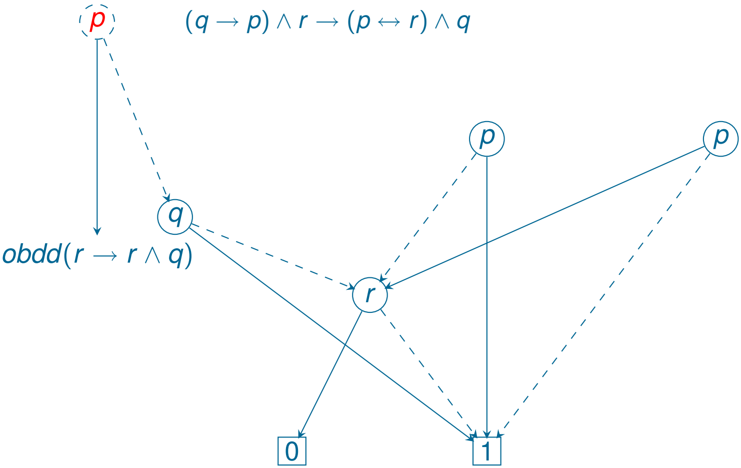

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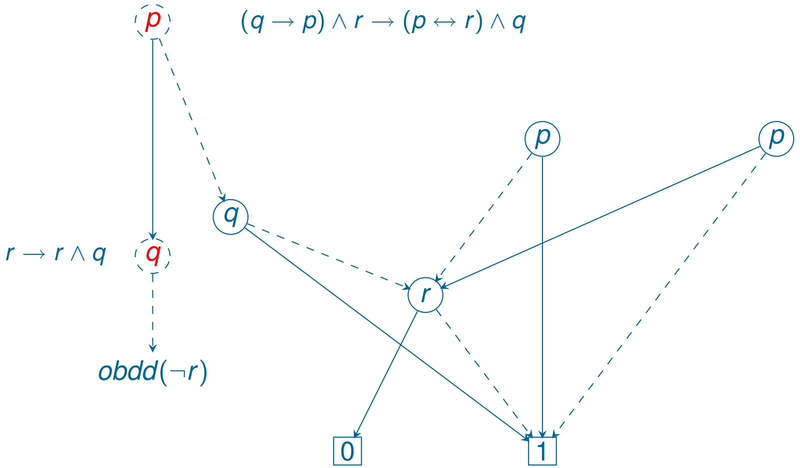

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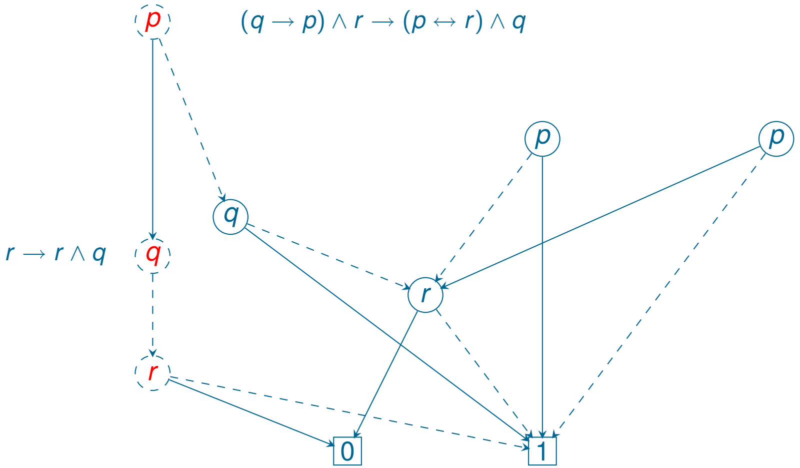

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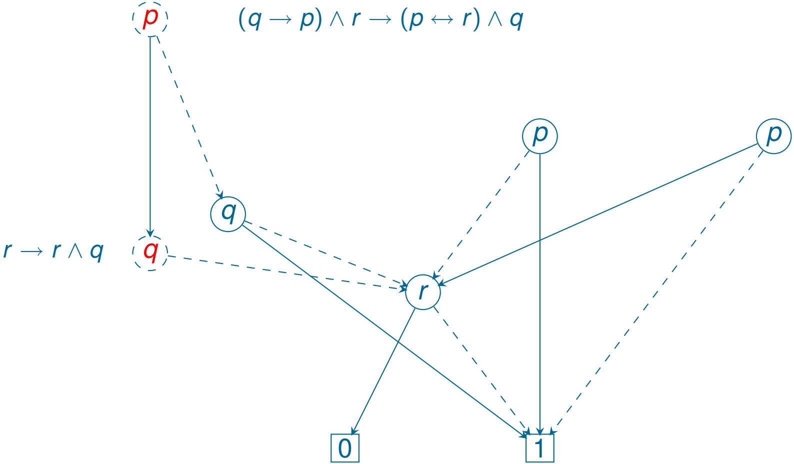

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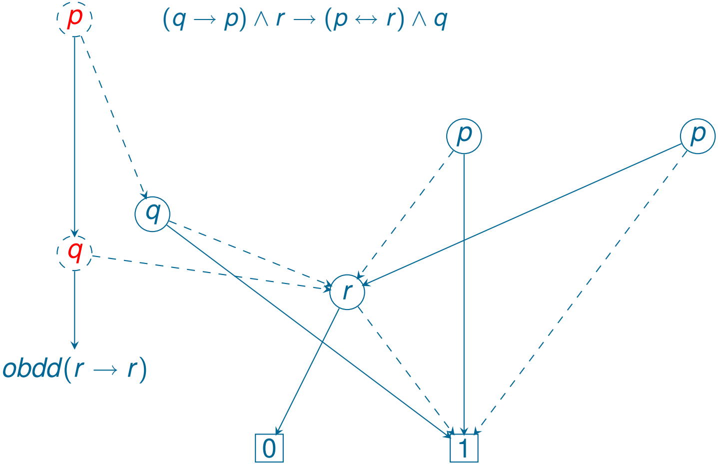
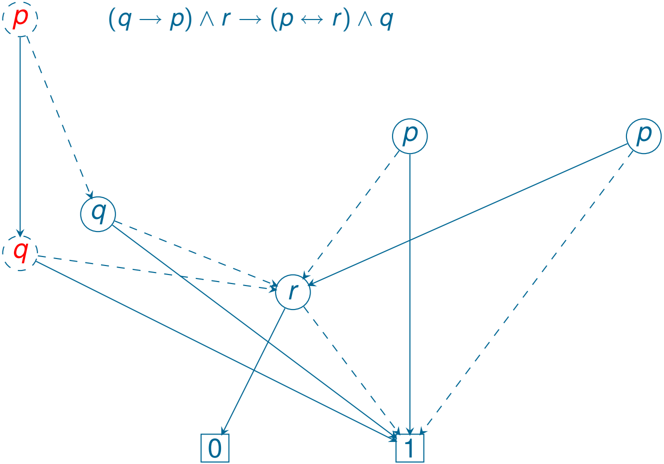


Example: Building OBDD

$$(q \to p) \land r \to (p \leftrightarrow r) \land q$$



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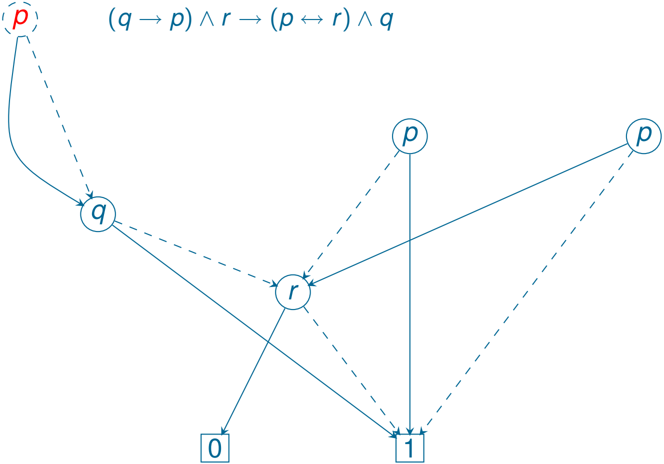
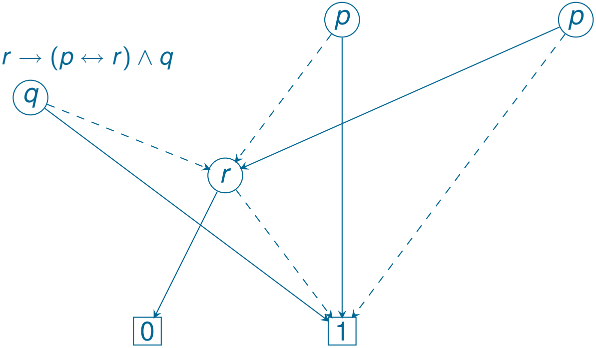


Example: Building OBDD

$(q \rightarrow p) \land r \rightarrow (p \leftrightarrow r) \land q$

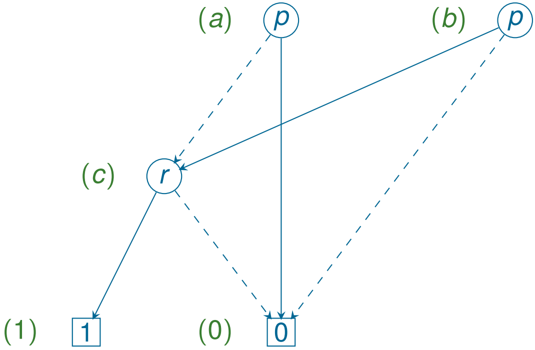


Disjunction

```
procedure disjunction(n_1, ..., n_m)
parameters: global dag D
input: nodes n_1, ..., n_m representing F_1, ..., F_m in D
output: a node n representing F_1 ∨ ... ∨ F_m in (modified) D
begin
  if some n_i is [1] then return [1]
  if m = 1 then return n_1
  if some n_i is [0] then
    return disjunction(n_1, ..., n_{i-1}, n_{i+1}, ..., n_m)
  p := max_atom(n_1, ..., n_m)
  forall i = 1 ... m
    if n_i is labelled by p
      then (l_i, r_i) := (neg(n_i), pos(n_i))
      else (l_i, r_i) := (n_i, n_i)
  k_1 := disjunction(l_1, ..., l_m)
  k_2 := disjunction(r_1, ..., r_m)
  return integrate(k_1, p, k_2, D)
end
```

Example: Disjunction

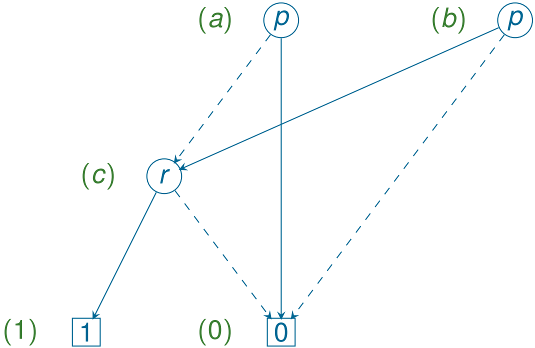
Computing $(\neg p \wedge r) \vee (p \wedge r)$:



Example: Disjunction

Computing $(\neg p \wedge r) \vee (p \wedge r)$:

dis(a,b)



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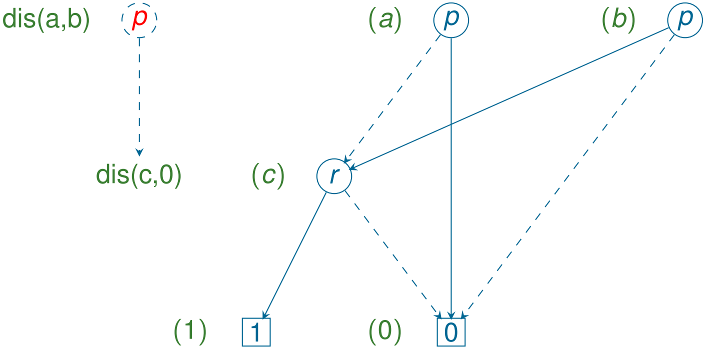

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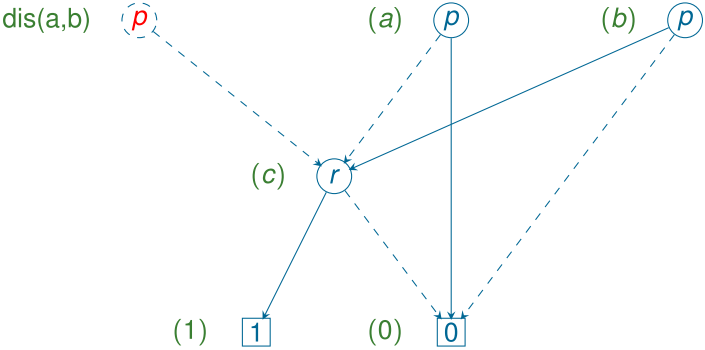

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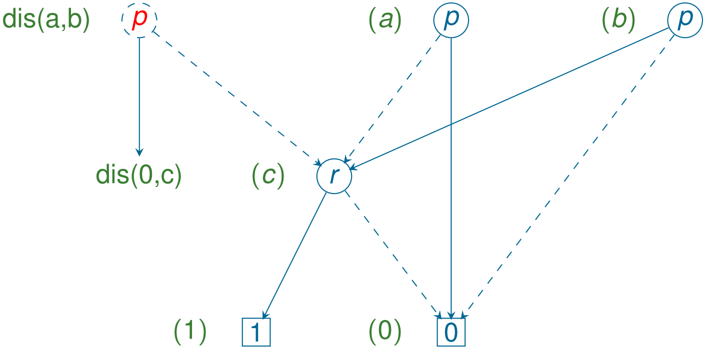
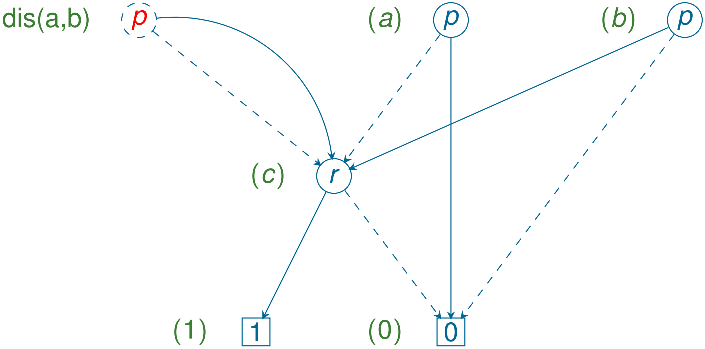


Example: Disjunction

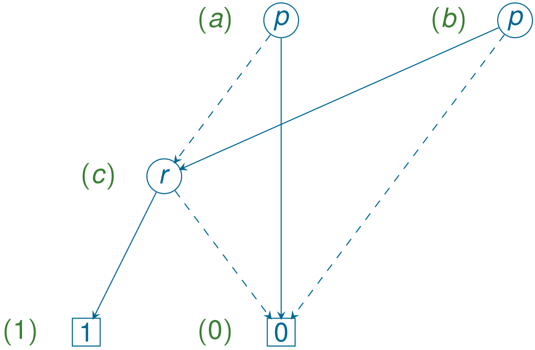
Computing $(\neg p \land r) \lor (p \land r)$:



Example: Disjunction

Computing $(\neg p \wedge r) \vee (p \wedge r)$:

dis(a,b) = c



Two-Player Games

Who is this man?

Two-Player Games

Does Garry Kasparov have a winning strategy?

Two-Player Games

Given a propositional formula G with variables $p_1, q_1, \ldots, p_n, q_n$.
There are two players: P and Q.
At step k each player makes a move:

1. the player P can choose a boolean value for the variable p_k;
2. the player Q can choose a boolean value for the variable q_k.

The player P wins if after n steps the chosen values make the formula G true.

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Winning Strategy

Problem: when does P have a winning strategy?
He has a winning strategy if

- there exists a move for P (a boolean value for for p_1) such that
- for all moves of Q (boolean values for for q_1)
- there exists a move for P (a boolean value for for p_2) such that
- for all moves of Q (boolean values for for q_2)
- ...
- for all moves of Q (boolean values for for q_n) the formula G becomes true.

The existence of a winning strategy can be expressed by a quantified boolean formula $\exists p_1 \forall q_1 \exists p_2 \forall q_2 \dots \exists p_n \forall q_n G$.

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Quantified Boolean Formulas

Propositional formula:

- Every boolean variable is a formula.
- $\top$ and $\bot$ are formulas.
- If $F_1, \ldots, F_n$ are formulas, where $n \geq 2$, then $(F_1 \wedge \ldots \wedge F_n)$ and $(F_1 \vee \ldots \vee F_n)$ are formulas.
- If F is a formula, then $\neg F$ is a formula.
- If F and G are formulas, then $(F \rightarrow G)$ and $(F \leftrightarrow G)$ are formulas.

Quantified boolean formulas:

- If p is a boolean variable and F is a formula, then $\forall pF$ and $\exists pF$ are formulas.

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Quantifiers

- $\forall$ is called the universal quantifier.
- $\exists$ is called the existential quantifier.
- Read $\forall p F$ as "for all p, F".
- Read $\exists p F$ as "there exists p such that F" or "for some p, F".

Semantics

1. $I(\top) = 1$ and $I(\bot) = 0$.
2. $I(F_1 \wedge \ldots \wedge F_n) = 1$ if and only if $I(F_i) = 1$ for all i.
3. $I(F_1 \vee \ldots \vee F_n) = 1$ if and only if $I(F_i) = 1$ for some i.
4. $I(\neg F) = 1$ if and only if $I(F) = 0$.
5. $I(F \rightarrow G) = 1$ if and only if $I(F) = 0$ or $I(G) = 1$.
6. $I(F \leftrightarrow G) = 1$ if and only if $I(F) = I(G)$.

Define

$$I[p \leftarrow b](q) \stackrel{\text{def}}{\Leftrightarrow} \begin{cases} I(q), & \text{if } p \neq q; \\ b, & \text{if } p = q. \end{cases}$$

7. $I(\forall p F) = 1$ if and only if $I[p \leftarrow 0](F) = 1$ and $I[p \leftarrow 1](F) = 1$.
8. $I(\exists p F) = 1$ if and only if $I[p \leftarrow 0](F) = 1$ or $I[p \leftarrow 1](F) = 1$.

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Evaluating a formula

Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$.

$$I_{10} \models \forall p \exists q (p \leftrightarrow q)$$

$\Leftrightarrow$

$$I_{00} \models \exists q (p \leftrightarrow q)$$
$$I_{10} \models \exists q (p \leftrightarrow q)$$

and

$\Leftrightarrow$

$$I_{00} \models p \leftrightarrow q$$
$$I_{01} \models p \leftrightarrow q$$

or

and

$$I_{10} \models p \leftrightarrow q$$
$$I_{11} \models p \leftrightarrow q$$

or

Evaluating a formula

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