Evaluating a formula

Let us evaluate $\forall p \exists q (p \leftrightarrow q)$ on the interpretation $\{p \mapsto 1, q \mapsto 0\}$.

Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$.

$$I_{10} \models \forall p \exists q (p \leftrightarrow q) \quad \Leftrightarrow \quad \begin{array}{l} I_{00} \models \exists q (p \leftrightarrow q) \\ I_{10} \models \exists q (p \leftrightarrow q) \end{array} \text{ and}$$

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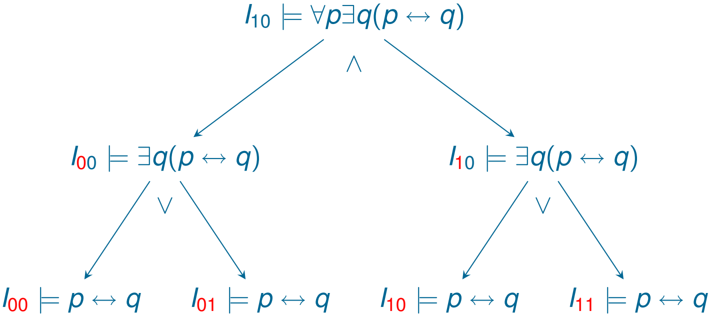

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Denote any interpretation $\{p \mapsto b_1, q \mapsto b_2\}$ by $I_{b_1 b_2}$. Use wildcards $\_$ to denote "any" boolean value.

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Subformula

Propositional formulas:

- The formulas $F_1, \dots, F_n$ are the immediate subformulas of the formulas $F_1 \wedge \dots \wedge F_n$ and $F_1 \vee \dots \vee F_n$.
- The formulas F is the immediate subformula of the formula $\neg F$.
- The formulas F_1, F_2 are the immediate subformulas of the formulas $F_1 \rightarrow F_2$ and $F_1 \leftrightarrow F_2$.
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Quantified boolean formulas:

- The formula F_1 is the immediate subformula of the formulas $\forall p F_1$ and $\exists p F_1$.

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Positions

Let $F|_\pi = G$.
Propositional formulas:

- If G has the form $G_1 \wedge \ldots \wedge G_n$ or $G_1 \vee \ldots \vee G_n$, then for all $i \in \{1, \ldots, n\}$ the position $\pi.i$ is a position in F and $pol(F, \pi.i) \overset{\text{def}}{\Leftrightarrow} pol(F, \pi)$.
- If G has the form $\neg G_1$, then $\pi.1$ is a position in F, $F|_{\pi.1} \overset{\text{def}}{\Leftrightarrow} G_1$ and $pol(F, \pi.1) \overset{\text{def}}{\Leftrightarrow} -pol(F, \pi)$.
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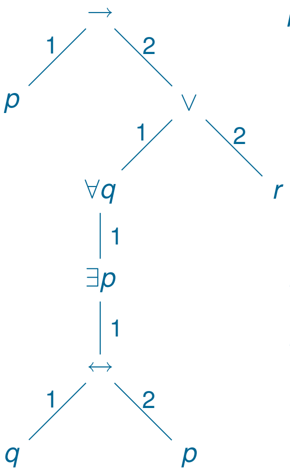

$p \rightarrow \forall q \exists p (q \leftrightarrow p) \vee r$

$F|_{2.1} = \forall q \exists p (q \leftrightarrow p)$

$F|_{2.1.1.1.1} = q$

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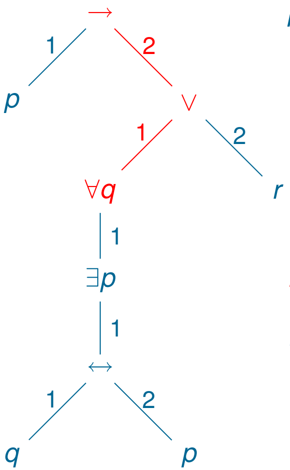

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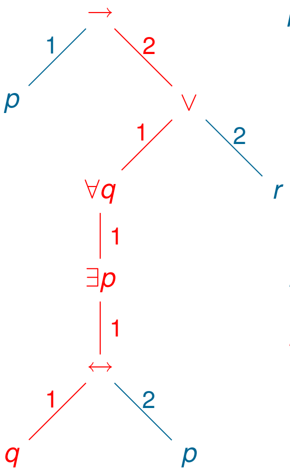

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Free and bound occurrences of variables

Let p be a boolean variable and $F|_\pi = p$.

- The occurrence of p at the position π in F is bound if π can be represented as a concatenation of two strings $\pi_1\pi_2$ such that $F|_{\pi_1}$ has the form $\forall pG$ or $\exists pG$ for some G.
  In other words, a bound occurence of p is an occurrence in the scope of $\forall p$ or $\exists p$.
- Free occurrence: not bound.
- Free (bound) variable of a formula: a variable with at least one free (bound) occurrence.
- Closed formula: formula with no free variables.

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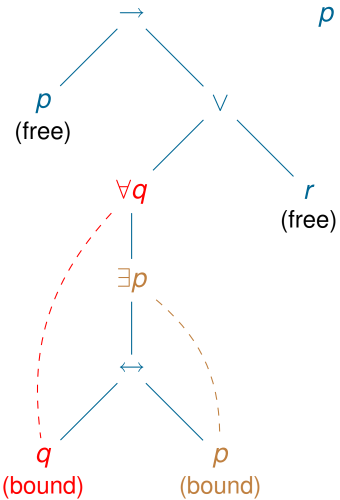
Example: Free and Bound Variables



$p \rightarrow \forall q \exists p (q \leftrightarrow p) \vee r$

Only Free Variables Matter

The truth value of a formula depends only on the truth values of free variables of the formula:

Lemma

Let for all free variables p of a formula F we have $I_1(p) = I_2(p)$. Then $I_1 \models F$ if and only if $I_2 \models F$.

Truth, Validity and Satisfiability

Validity and satisfiability are defined as for propositional formulas.

There is no difference between these notions for closed formulas:

Lemma

For every interpretation I and closed formula F the following propositions are equivalent: (i) $I \models F$; (ii) F is satisfiable; and (iii) F is valid.

Validity and satisfiability can be expressed through truth:

Lemma

Let F be a formula with free variables $p_1, \ldots, p_n$.

- *F is satisfiable if and only if the formula $\exists p_1 \ldots \exists p_n F$ is satisfiable (true, valid).*
- *F is valid if and only if the formula $\forall p_1 \ldots \forall p_n F$ is valid (true, satisfiable).*

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Equivalent replacement

Lemma

Let I *be an interpretation and* $I \models F_1 \leftrightarrow F_2$. *Then* $I \models G[F_1] \leftrightarrow G[F_2]$.

Theorem (Equivalent Replacement)

Let $F_1 \equiv F_2$. *Then* $G[F_1] \equiv G[F_2]$.

Substitutions for propositional formulas

Substitution: $(F)_p^G$: denotes the formula obtained from F by replacing all occurrences of the variable p by G.

Example: $((p \vee s) \wedge (q \rightarrow p))_p^{(l \wedge s)} =$
$(((l \wedge s) \vee s) \wedge (q \rightarrow (l \wedge s)))$

Properties: If we apply any substitution to a valid formula then we also obtain a valid formula.

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Substitution for quantified formulas

Some problems...

Consider $\exists q(\neg p \leftrightarrow q)$.

We cannot simply replace variables by formulas any more:
$\exists(r \rightarrow r)(\neg p \leftrightarrow r \rightarrow r)$???

Free variables are parameters: we can only substitute for parameters.
But a variable can have both free and bound occurrences in a formula, e.g. $(\forall p p \rightarrow q) \wedge (q \vee (q \rightarrow p))$

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Renaming bound variables

Notation: $\exists\!\!\!\forall$: any of $\exists$, $\forall$ and $\wedge\!\!\!\vee$: any of $\wedge$, $\vee$.

Renaming bound variables in F:
Let $F[\ \exists\!\!\!\forall pG\ ]$.

1. Take a fresh variable q (that is a variable not occurring in F);
2. Replace all free occurrences of p in G (note: not in F!) by q obtaining G'.
3. So we obtain the $F[\ \exists\!\!\!\forall qG'\ ]$ as the result.

Lemma

$F[\ \exists\!\!\!\forall pG\ ] \equiv F[\ \exists\!\!\!\forall qG'\ ]$

Example:
$\exists q(\ \forall p((p \to q) \wedge p)\ ) \vee p.$

Then we can rename p into r obtaining
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End of Lecture 13

Slides for lecture 13 ended here . . .