Literal, clause

- Literal: either an atom p (positive literal) or its negation $\neg p$ (negative literal).
- The complementary literal to L:

$$\overline{L} \overset{\text{def}}{\Leftrightarrow} \begin{cases} \neg L, & \text{if } L \text{ is positive;} \\ p, & \text{if } L \text{ has the form } \neg p. \end{cases}$$

  In other words, p and $\neg p$ are complementary.
- Clause: a disjunction $L_1 \vee \ldots \vee L_n$, $n \geq 0$ of literals.
  - Empty clause, denoted by $\Box$: $n = 0$ (the empty clause is false in every interpretation).
  - Unit clause: $n = 1$.
  - Horn clause: a clause with at most one positive literal.

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CNF

- A formula A is in conjunctive normal form, or simply CNF, if it is either $\top$, or $\bot$, or a conjunction of disjunctions of literals:

$$A = \bigwedge_i \bigvee_j L_{i,j}.$$

  (That is, a conjunction of clauses.)
- A formula B is called a conjunctive normal form of a formula A if B is equivalent to A and B is in conjunctive normal form.

Satisfiability on CNF

- An interpretation I satisfies a formula in CNF

$$A = \bigwedge_i \bigvee_j L_{i,j}.$$

if and only if it satisfies every clause

$$\bigvee_j L_{i,j}.$$

in it.

- An interpretation I satisfies a clause

$$L_1 \vee \ldots \vee L_k$$

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CNF transformation

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\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), \\
A \rightarrow B & \Rightarrow & \neg A \vee B, \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) \quad \wedge \\
 & & \ldots \quad \wedge \\
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\end{array}
$$

A formula to which no rewrite rule is applicable

- contains no $\leftrightarrow$;
- contains no $\rightarrow$;
- may only contain $\neg$ applied to atoms;
- cannot contain $\wedge$ in the scope of $\vee$;
- (hence) is in CNF.

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\neg\neg A & \Rightarrow & A, \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & \begin{array}{cc} (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\ \ldots & \wedge \\ (A_m \vee B_1 \vee \ldots \vee B_n). & \end{array}
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CNF, example

$$
\begin{aligned}
&\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r)) \Rightarrow \\
&\neg(\neg((p \to q) \wedge (p \wedge q \to r)) \vee (p \to r)) \Rightarrow \\
&\neg\neg((p \to q) \wedge (p \wedge q \to r)) \wedge \neg(p \to r) \Rightarrow \\
&(p \to q) \wedge (p \wedge q \to r) \wedge \neg(p \to r) \Rightarrow \\
&(p \to q) \wedge (p \wedge q \to r) \wedge \neg(\neg p \vee r) \Rightarrow \\
&(p \to q) \wedge (p \wedge q \to r) \wedge \neg\neg p \wedge r \Rightarrow \\
&(p \to q) \wedge (p \wedge q \to r) \wedge p \wedge \neg r \Rightarrow \\
&(p \to q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow \\
&(p \to q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \\
&(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r
\end{aligned}
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$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \to B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
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\end{array}
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CNF, example

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow$$
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A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), & \\
A \to B & \Rightarrow & \neg A \lor B, & \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, & \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) & \land \\
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\end{array}$$

CNF, example

$$\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \wedge (p \wedge q \to r)) \vee (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \wedge (p \wedge q \to r)) \wedge \neg(p \to r) \Rightarrow$$
$$(p \to q) \wedge (p \wedge q \to r) \wedge \neg(p \to r) \Rightarrow$$
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$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
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\neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge r \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \\
(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r
\end{array}
$$

$$
\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), \\
A \rightarrow B & \Rightarrow & \neg A \vee B, \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) \quad \wedge \\
 & & \ldots \quad \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n).
\end{array}
$$

CNF, example

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \land q) \lor r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

$$\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), & \\
A \to B & \Rightarrow & \neg A \lor B, & \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, & \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) & \land \\
& & \cdots & \land \\
& & (A_m \lor B_1 \lor \ldots \lor B_n). &
\end{array}$$

CNF, example

$$
\begin{array}{l}
\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow \\
\neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow \\
\neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge r \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \\
(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r
\end{array}
$$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \rightarrow B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}
$$

CNF, example

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow$$
$$\neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow$$
$$\neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge r \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$
$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

$$\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \rightarrow B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \ldots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}$$

CNF, example

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow$$
$$\neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow$$
$$\neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge r \Rightarrow$$
$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow$$
$$(p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$
$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

$$
\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), \\
A \rightarrow B & \Rightarrow & \neg A \vee B, \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) \quad \wedge \\
 & & \ldots \quad \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n).
\end{array}
$$

CNF, example

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \land q) \lor r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), \\
A \to B & \Rightarrow & \neg A \lor B, \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) \quad \land \\
 & & \ldots \quad \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n).
\end{array}$$

CNF, example

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \land q) \lor r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), \\
A \to B & \Rightarrow & \neg A \lor B, \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) \quad \land \\
& & \ldots \quad \land \\
& & (A_m \lor B_1 \lor \ldots \lor B_n).
\end{array}$$

CNF, example

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \land q) \lor r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

$$\begin{array}{rcl} A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), \\ A \to B & \Rightarrow & \neg A \lor B, \\ \neg(A \land B) & \Rightarrow & \neg A \lor \neg B, \\ \neg(A \lor B) & \Rightarrow & \neg A \land \neg B, \\ \neg\neg A & \Rightarrow & A, \\ (A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) \quad \land \\ & & \ldots \quad \land \\ & & (A_m \lor B_1 \lor \ldots \lor B_n). \end{array}$$

CNF, example

$$
\begin{array}{l}
\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow \\
\neg(\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \vee (p \rightarrow r)) \Rightarrow \\
\neg\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r)) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(p \rightarrow r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg(\neg p \vee r) \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge \neg\neg p \wedge r \Rightarrow \\
(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge (\neg(p \wedge q) \vee r) \wedge p \wedge \neg r \Rightarrow \\
(p \rightarrow q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r \\
(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r
\end{array}
$$

$$
\begin{array}{rcll}
A \leftrightarrow B & \Rightarrow & (\neg A \vee B) \wedge (\neg B \vee A), & \\
A \rightarrow B & \Rightarrow & \neg A \vee B, & \\
\neg(A \wedge B) & \Rightarrow & \neg A \vee \neg B, & \\
\neg(A \vee B) & \Rightarrow & \neg A \wedge \neg B, & \\
\neg\neg A & \Rightarrow & A, & \\
(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n & \Rightarrow & (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\
 & & \cdots & \wedge \\
 & & (A_m \vee B_1 \vee \ldots \vee B_n). &
\end{array}
$$

CNF, example

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\neg(\neg((p \to q) \land (p \land q \to r)) \lor (p \to r)) \Rightarrow$$
$$\neg\neg((p \to q) \land (p \land q \to r)) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(p \to r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg(\neg p \lor r) \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land \neg\neg p \land r \Rightarrow$$
$$(p \to q) \land (p \land q \to r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg(p \land q) \lor r) \land p \land \neg r \Rightarrow$$
$$(p \to q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

$$\begin{array}{rcl}
A \leftrightarrow B & \Rightarrow & (\neg A \lor B) \land (\neg B \lor A), \\
A \to B & \Rightarrow & \neg A \lor B, \\
\neg(A \land B) & \Rightarrow & \neg A \lor \neg B, \\
\neg(A \lor B) & \Rightarrow & \neg A \land \neg B, \\
\neg\neg A & \Rightarrow & A, \\
(A_1 \land \ldots \land A_m) \lor B_1 \lor \ldots \lor B_n & \Rightarrow & (A_1 \lor B_1 \lor \ldots \lor B_n) \quad \land \\
 & & \ldots \quad \land \\
 & & (A_m \lor B_1 \lor \ldots \lor B_n).
\end{array}$$

CNF and satisfiability

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\dots$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

Therefore, the formula

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r))$$

has the same models as the set consisting of four clauses

$$\neg p \lor q$$
$$\neg p \lor \neg q \lor r$$
$$p$$
$$\neg r$$

The CNF transformation allows one to reduce the satisfiability problem for formulas to the satisfiability problem for sets of clauses.

CNF and satisfiability

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r)) \Rightarrow$$
$$\cdots$$
$$(\neg p \lor q) \land (\neg p \lor \neg q \lor r) \land p \land \neg r$$

Therefore, the formula

$$\neg((p \to q) \land (p \land q \to r) \to (p \to r))$$

has the same models as the set consisting of four clauses

$$\begin{array}{l} \neg p \lor q \\ \neg p \lor \neg q \lor r \\ p \\ \neg r \end{array}$$

The CNF transformation allows one to reduce the satisfiability problem for formulas to the satisfiability problem for sets of clauses.

CNF and satisfiability

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)) \Rightarrow$$
$$\cdots$$
$$(\neg p \vee q) \wedge (\neg p \vee \neg q \vee r) \wedge p \wedge \neg r$$

Therefore, the formula

$$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$$

has the same models as the set consisting of four clauses

$$\begin{array}{l} \neg p \vee q \\ \neg p \vee \neg q \vee r \\ p \\ \neg r \end{array}$$

The CNF transformation allows one to reduce the satisfiability problem for formulas to the satisfiability problem for sets of clauses.

Problem

Compute the CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))).$$

$$
\begin{array}{l}
p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \Rightarrow \\
(\neg p_1 \vee (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))\wedge \\
(p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \\
(\neg p_1 \vee \quad ((\neg p_2 \vee (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))\wedge \\
\qquad\qquad (p_2 \vee \neg(p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))\wedge \\
(p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))
\end{array}
$$

If we continue, the formula will grow exponentially.

Problem

Compute the CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))).$$

$$\begin{aligned} & p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \Rightarrow \\ & (\neg p_1 \vee (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))\wedge \\ & (p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \Rightarrow \\ & (\neg p_1 \vee \quad ((\neg p_2 \vee (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))\wedge \\ & \qquad\quad (p_2 \vee \neg(p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))))\wedge \\ & (p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \end{aligned}$$

If we continue, the formula will grow exponentially.

Problem

Compute the CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))).$$

$$\begin{aligned}
& p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \Rightarrow \\
& (\neg p_1 \lor (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \land \\
& (p_1 \lor \neg (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \Rightarrow \\
& (\neg p_1 \lor \quad ((\neg p_2 \lor (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \land \\
& \qquad\qquad (p_2 \lor \neg (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))) \land \\
& (p_1 \lor \neg (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))
\end{aligned}$$

If we continue, the formula will grow exponentially.

Problem

Compute the CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))).$$

$$\begin{array}{l}
p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \Rightarrow \\
(\neg p_1 \vee (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \wedge \\
(p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \Rightarrow \\
(\neg p_1 \vee \quad ((\neg p_2 \vee (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))\wedge \\
\qquad\qquad (p_2 \vee \neg(p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))))\wedge \\
(p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))
\end{array}$$

If we continue, the formula will grow exponentially.

Problem

Compute the CNF of

$$p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))).$$

$$\begin{aligned}
& p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \Rightarrow \\
& (\neg p_1 \vee (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \wedge \\
& (p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))) \Rightarrow \\
& (\neg p_1 \vee \quad ((\neg p_2 \vee (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))\wedge \\
& \qquad\qquad (p_2 \vee \neg(p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))))\wedge \\
& (p_1 \vee \neg(p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))))
\end{aligned}$$

If we continue, the formula will grow exponentially.

CNF is exponential

There are formulas for which the shortest CNF has exponential size.

Is there any way to avoid exponential blowup?

CNF is exponential

There are formulas for which the shortest CNF has exponential size.

Is there any way to avoid exponential blowup?

Idea

Using so-called naming or definition introduction.

- Take a non-trivial subformula A.
- Introduce a new name n for it. A name is a new propositional variable.
- Add a formula stating that n is equivalent to A (definition for n).

$$
\begin{array}{l}
p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \\
n \leftrightarrow (p_5 \leftrightarrow p_6)
\end{array}
$$

- Replace the subformula by its name:

$$
\begin{array}{l}
p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow n))) \\
n \leftrightarrow (p_5 \leftrightarrow p_6)
\end{array}
$$

The new set of two formulas has the same models as the original one if we restrict ourselves to the original set of variables $\{p_1, \ldots, p_6\}$.
But this set is not equivalent to the original formula.

Idea

Using so-called naming or definition introduction.

- Take a non-trivial subformula A.
- Introduce a new name n for it. A name is a new propositional variable.
- Add a formula stating that n is equivalent to A (definition for n).

$$
\begin{array}{l}
p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6)))) \\
n \leftrightarrow (p_5 \leftrightarrow p_6)
\end{array}
$$

- Replace the subformula by its name:

$$
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p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow n))) \\
n \leftrightarrow (p_5 \leftrightarrow p_6)
\end{array}
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The new set of two formulas has the same models as the original one if we restrict ourselves to the original set of variables $\{p_1, \ldots, p_6\}$. But this set is not equivalent to the original formula.

Idea

Using so-called naming or definition introduction.

- Take a non-trivial subformula A.
- Introduce a new name n for it. A name is a new propositional variable.
- Add a formula stating that n is equivalent to A (definition for n).

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After several steps

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$$\begin{aligned} p_1 &\leftrightarrow (p_2 \leftrightarrow n_3); \\ n_3 &\leftrightarrow (p_3 \leftrightarrow n_4); \\ n_4 &\leftrightarrow (p_4 \leftrightarrow n_5); \\ n_5 &\leftrightarrow (p_5 \leftrightarrow p_6). \end{aligned}$$

The conversion of the original formula to CNF introduces 32 copies of p_6.

The conversion of the new set of formulas to CNF introduces 4 copies of p_6.

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Clausal Form

- Clausal form of a formula A: a set of clauses which is satisfiable if and only if A is satisfiable.
- Clausal form of a set S of formulas: a set of clauses which is satisfiable if and only if so is S.

We can require even more: that A and S have the same models in the language of A.

Using clausal normal forms instead of conjunctive normal forms we can convert any formula to a set of clauses in almost linear time.

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Definitional Clause Form Transformation

This algorithm converts a formula A into a set of clauses S such that S is a clausal normal form of A.

If A has the form $C_1 \wedge \ldots \wedge C_n$, where $n \geq 1$ and each C_i is a clause, then $S \stackrel{\text{def}}{\Leftrightarrow} \{C_1, \ldots, C_n\}$.

Otherwise, introduce a name for each subformula B of A such that B is not a literal and use this name instead of the formula.

Example

	subformula	definition	clauses
			n_1
n_1	$\neg((p \to q) \land (p \land q \to r) \to (p \to r))$	$n_1 \leftrightarrow \neg n_2$	$\neg n_1 \lor \neg n_2$ $n_1 \lor n_2$
n_2	$(p \to q) \land (p \land q \to r) \to (p \to r)$	$n_2 \leftrightarrow (n_3 \to n_7)$	$\neg n_2 \lor \neg n_3 \lor n_7$ $n_3 \lor n_2$ $\neg n_7 \lor n_2$
n_3	$(p \to q) \land (p \land q \to r)$	$n_3 \leftrightarrow (n_4 \land n_5)$	$\neg n_3 \lor n_4$ $\neg n_3 \lor n_5$ $\neg n_4 \lor \neg n_5 \lor n_3$
n_4	$p \to q$	$n_4 \leftrightarrow (p \to q)$	$\neg n_4 \lor \neg p \lor q$ $p \lor n_4$ $\neg q \lor n_4$
n_5	$p \land q \to r$	$n_5 \leftrightarrow (n_6 \to r)$	$\neg n_5 \lor \neg n_6 \lor r$ $n_6 \lor n_5$ $\neg r \lor n_5$
n_6	$p \land q$	$n_6 \leftrightarrow (p \land q)$	$\neg n_6 \lor p$ $\neg n_6 \lor q$ $\neg p \lor \neg q \lor n_6$
n_7	$p \to r$	$n_7 \leftrightarrow (p \to r)$	$\neg n_7 \lor \neg p \lor r$ $p \lor n_7$ $\neg r \lor n_7$

Exercise 1 (deadline: October 9th, 4pm)

Exercise 3.1

The following formula has its parentheses removed. Restore the parentheses.

$$\neg p_1 \to \neg\neg p_2 \leftrightarrow p_3 \wedge p_4.$$

Exercise 3.9

Show that the formulas $p \to (q \to r)$ and $(p \to q) \to r$ are not equivalent by finding an interpretation in which they have different truth values.

Exercise 2 (deadline: October 16th, 4pm)

Exercise 4.2 (b)

Build a truth table for the following formula

$$p \leftrightarrow (\neg r \rightarrow \neg p).$$

Exercise 4.5

Check, using splitting, whether the formula $(p \leftrightarrow q) \wedge ((p \wedge \neg q) \vee (q \wedge \neg p))$ is satisfiable. Split on the atom p first.

Exercise 5.4 (a)

Apply the standard CNF transformation algorithm to the following formula:

$$\neg((p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p))$$