# Positions and Subformulas

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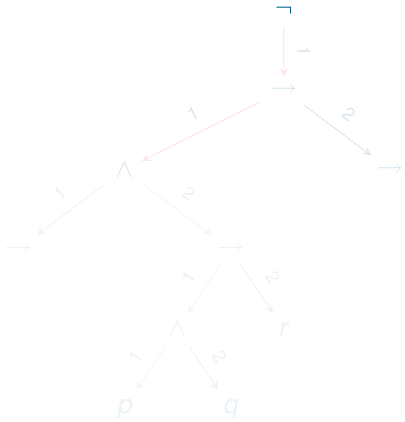

$A|_\epsilon = \neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r))$.

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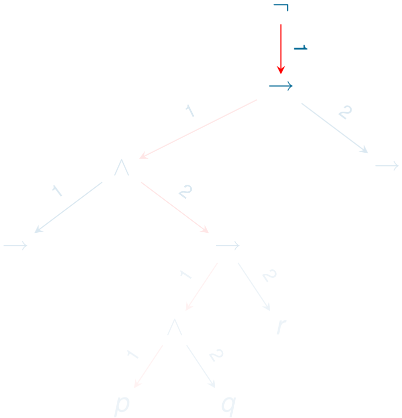

$A|_1 = (p \to q) \wedge (p \wedge q \to r) \to (p \to r)$.

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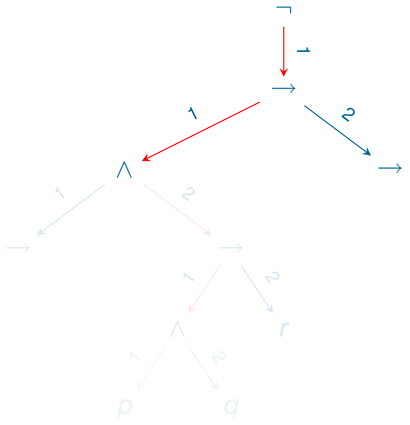


$A|_{11} = (p \to q) \land (p \land q \to r)$.

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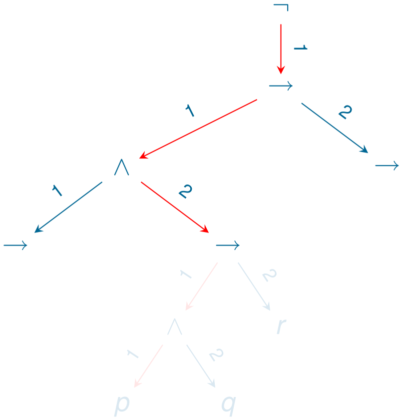


$A|_{112} = p \wedge q \to r$.

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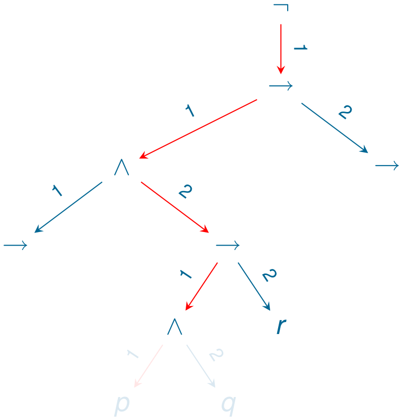


$A|_{1121} = p \wedge q$.

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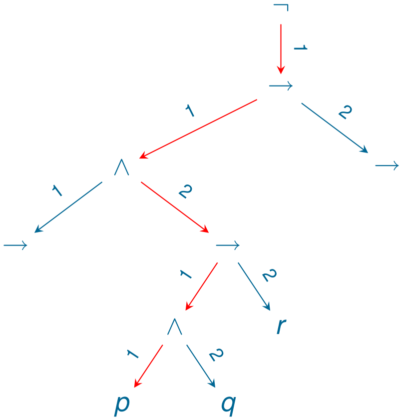
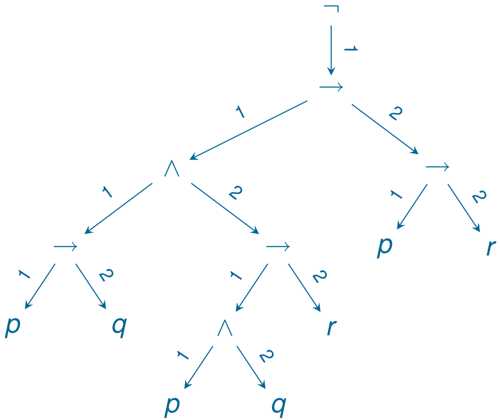



$A|_{11211} = p$.

# All positions in this formula

$$A \overset{\text{def}}{\Leftrightarrow} \neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)).$$



# Polarity

Polarity of subformula at a position. Notation: $pol(A, \pi)$.

1. For every formula $A$, $\epsilon$ is a position in $A$, $A|_\epsilon \stackrel{\text{def}}{\Leftrightarrow} A$ and $pol(A, \epsilon) \stackrel{\text{def}}{\Leftrightarrow} 1$;
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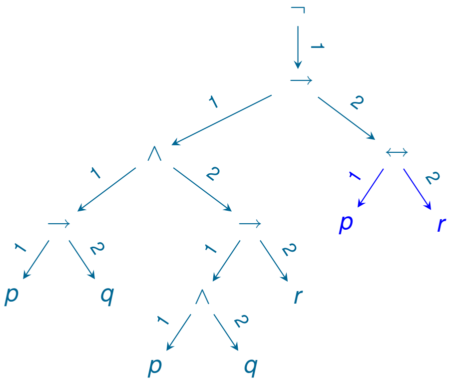
# The coloring algorithm for determining polarity

$\neg((p \to q) \land (p \land q \to r) \to (p \leftrightarrow (r \to q)))$.

- Color in blue all arcs below an equivalence.
- Color in red all arcs going down from a negation or left-hand side of an implication.

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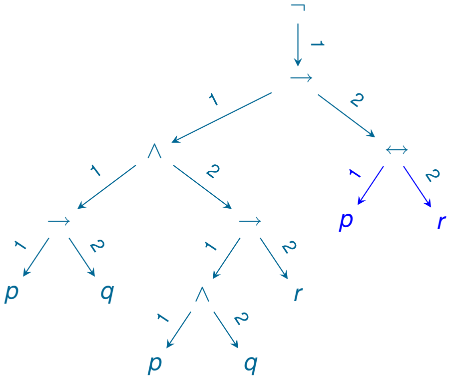


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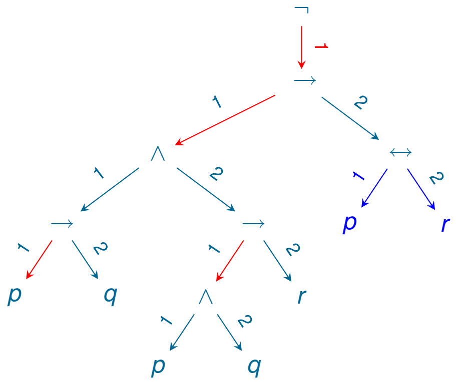


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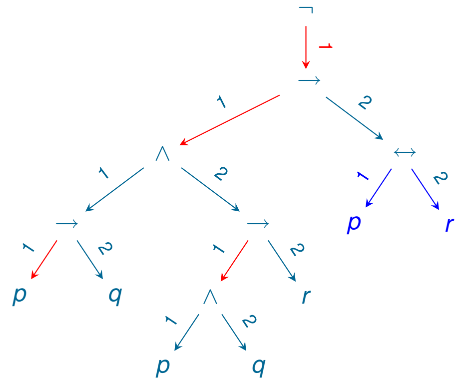


- If a position has at least one blue arc above it, its polarity is 0.
- Otherwise, its polarity is $-1$ if it has an odd number of red arcs above it.

# The coloring algorithm for determining polarity

$\neg((p \to q) \land (p \land q \to r) \to (p \leftrightarrow (r \to q)))$.

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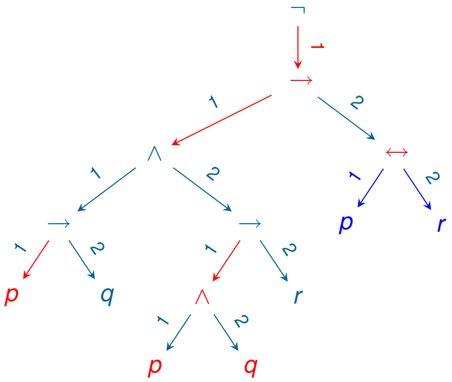


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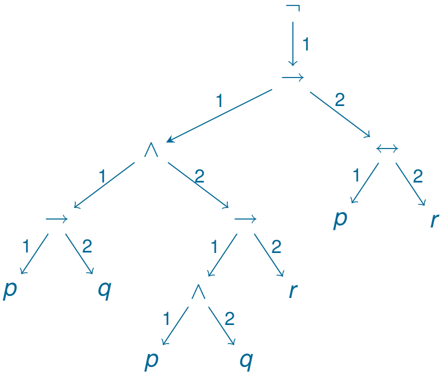

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# The coloring algorithm for determining polarity

$\neg((p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \leftrightarrow (r \rightarrow q)))$.

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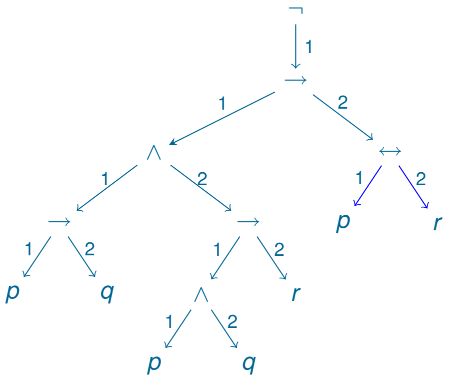


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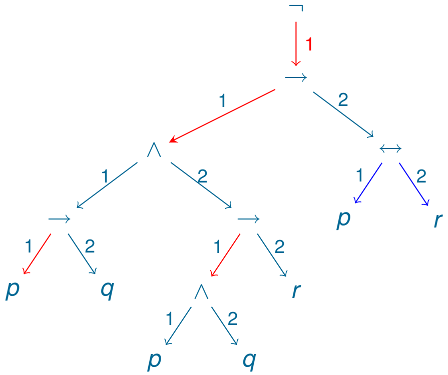


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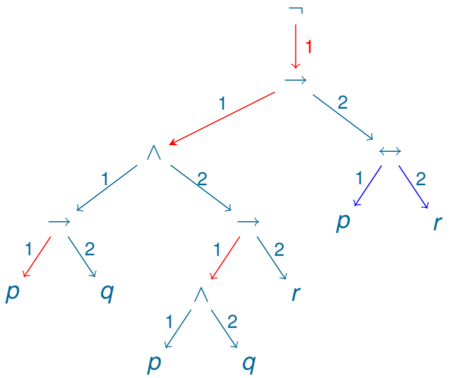


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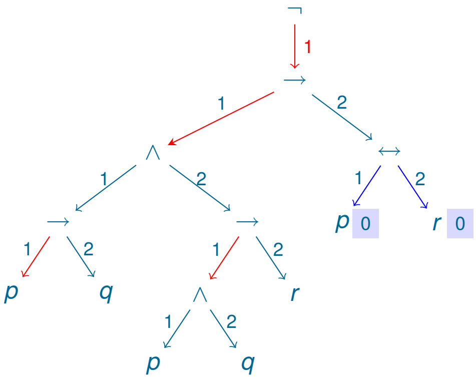


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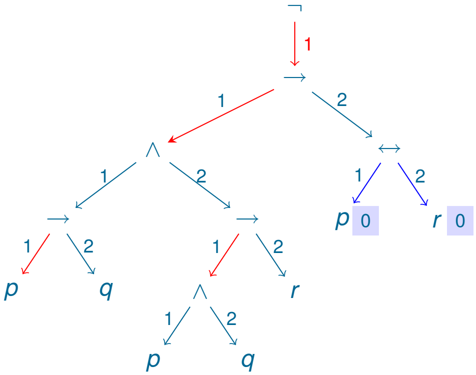


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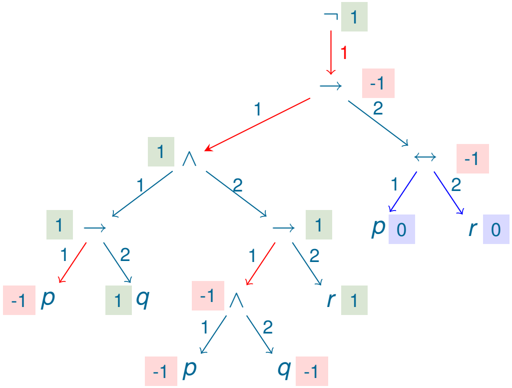


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# Position and polarity, again

| position | subformula | polarity |
|---|---|---|
| $\epsilon$ | $\neg((p \to q) \wedge (p \wedge q \to r) \to (p \to r))$ | 1 |
| 1 | $(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$ | $-1$ |
| 1.1 | $(p \to q) \wedge (p \wedge q \to r)$ | 1 |
| 1.1.1 | $p \to q$ | 1 |
| 1.1.1.1 | $p$ | $-1$ |
| 1.1.1.2 | $q$ | 1 |
| 1.1.2 | $p \wedge q \to r$ | 1 |
| 1.1.2.1 | $p \wedge q$ | $-1$ |
| 1.1.2.1.1 | $p$ | $-1$ |
| 1.1.2.1.2 | $q$ | $-1$ |
| 1.1.2.2 | $r$ | 1 |
| 1.2 | $p \to r$ | $-1$ |
| 1.2.1 | $p$ | 1 |
| 1.2.2 | $r$ | $-1$ |

# Monotonic replacement

Notation: $A[B]_\pi$:

- formula $A$ with the subformula $B$ at the position $\pi$;
- formula $A$ with the subformula at the position $\pi$ replaced by $B$.

### Lemma (Monotonic Replacement)

*Let $A, B, B'$ be formulas, $I$ be an interpretation, and $I \models B \rightarrow B'$. If $pol(A, \pi) = 1$, then $I \models A[B]_\pi \rightarrow A[B']_\pi$. Likewise, if $pol(A, \pi) = -1$, then $I \models A[B']_\pi \rightarrow A[B]_\pi$.*

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# Optimised Definitional Clause Form Transformation

If we introduce a name for a subformula and the occurence of the subformula is positive or negative, then an implication is used instead of equivalence.

See Chapter 5 for a precise description!

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# Example

| | subformula | definition | clauses |
|---|---|---|---|
| | | | $n_1$ |
| $n_1$ | $\neg((p \to q) \land (p \land q \to r) \to (p \to r))$ | $n_1 \to \neg n_2$ | $\neg n_1 \lor \neg n_2$ |
| | | | $n_1 \lor n_2$ |
| $n_2$ | $(p \to q) \land (p \land q \to r) \to (p \to r)$ | $(n_3 \to n_7) \to n_2$ | $\neg n_2 \lor \neg n_3 \lor n_7$ |
| | | | $n_3 \lor n_2$ |
| | | | $\neg n_7 \lor n_2$ |
| $n_3$ | $(p \to q) \land (p \land q \to r)$ | $n_3 \to (n_4 \land n_5)$ | $\neg n_3 \lor n_4$ |
| | | | $\neg n_3 \lor n_5$ |
| | | | $\neg n_4 \lor \neg n_5 \lor n_3$ |
| $n_4$ | $p \to q$ | $n_4 \to (p \to q)$ | $\neg n_4 \lor \neg p \lor q$ |
| | | | $p \lor n_4$ |
| | | | $\neg q \lor n_4$ |
| $n_5$ | $p \land q \to r$ | $n_5 \to (n_6 \to r)$ | $\neg n_5 \lor \neg n_6 \lor r$ |
| | | | $n_6 \lor n_5$ |
| | | | $\neg r \lor n_5$ |
| $n_6$ | $p \land q$ | $n_6 \to (p \land q)$ | $\neg n_6 \lor p$ |
| | | | $\neg n_6 \lor q$ |
| | | | $\neg p \lor \neg q \lor n_6$ |
| $n_7$ | $p \to r$ | $(p \to r) \to n_7$ | $\neg n_7 \lor \neg p \lor r$ |
| | | | $p \lor n_7$ |
| | | | $\neg r \lor n_7$ |

# End of Lecture 5

Slides for lecture 5 ended here . . .